Interger approximations of equilateral triangles

Douglas Wilhelm Harder

July 26, 2026

In creating an approximation of an equilateral triangle where the points fall on the points on an integer grid, it is necessary to find points $(-m, 0)$, $(0, n)$ and $(m, 0)$ where $\sqrt{m^2 + n^2} \approx 2m$. If you experimentally (in this case, using Maple) find ever increasing values of m such that $\frac{\text{round}(\sqrt{3}m)}{m}$ is a better approximation of $\sqrt{3}$ than the previous smaller values of m, we get

$$1, 2, 3, 4, 11, 26, 41, 56, 153, 362, 571, 780, 2131, 5042, 7953, 10864, 29681, 70226,$$

$$110771, 151316, 413403, 978122, 1542841, 2107560, 5757961, \ldots,$$

while the sequence of the n values are found by calculating $\text{round}(\sqrt{3}m)$, yielding

$$2, 3, 5, 7, 19, 45, 71, 97, 265, 627, 989, 1351, 3691, 8733, 13775, 18817, 51409, 121635,$$

$$191861, 262087, 716035, 1694157, 2672279, 3650401, 9973081, \ldots.$$

What is fascinating is that the error with each successive approximation drops by a factor that, in the limit, appears to be approach $2 + \sqrt{3}$. Not only that, but the ratio between the successive values of m appear to approach a pattern that cycles, so that if we define

$$c_0 \leftarrow \frac{3}{2} + \frac{\sqrt{3}}{2}, c_1 \leftarrow 1 + \frac{\sqrt{3}}{3}, c_2 \leftarrow \frac{1}{2} + \frac{\sqrt{3}}{2}, c_3 \leftarrow 1 + \sqrt{3},$$

as well as $m_0 \leftarrow 1$, and then define $m_{k+1} \leftarrow \text{round}(c_{k \bmod 4} m_k)$ we can reproduce the above numbers exactly, and the tables below calculate successively larger values of these pairs.

To note, the product $c_0 c_1 c_2 c_3 = 7 + 4\sqrt{2}$ and the geometric mean of these four is $\frac{1+\sqrt{3}}{\sqrt{2}} \approx 1.93$.

One oddity is that neither of the above sequences appears in the *The On-Line Encyclopedia of Integer Sequences* (OEIS).

Motivation: In many cases, as one finds better and better rational approximations of various real numbers, one usually finds one small ratio where the error drops significantly where all other approximations with smaller denominators have significantly larger errors. For example, while $\frac{22}{7}$ with a percent relative error of 0.04%

is an "okay" approximation of π (one part in 2500), but $\frac{355}{113}$ is significantly better than all previous approximations by a significant margin with a percent relative error of 0.0000085%, or one part in almost twelve million. While trying to find *nice* integer grid approximations of equilateral triangles, no such jump ever occurred, so I investigated further. This is related to continued fractions, but the continued fraction approximation $\sqrt{3}$ alternates between 1 and 2, and only every second value is what we see above, while the alternating values only match the above if the last 1 is turned into a 2. That is, the general continued fraction approximation is

$$1 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \cdots}}}}}$$

but the best approximations above are not exactly matching these.

k	m_k	n_k
0	1	2
1	2	3
2	3	5
3	4	7
4	11	19
5	26	45
6	41	71
7	56	97
8	153	265
9	362	627
10	571	989
11	780	1351
12	2131	3691
13	5042	8733
14	7953	13775
15	10864	18817
16	29681	51409
17	70226	121635
18	110771	191861
19	151316	262087
20	413403	716035
21	978122	1694157
22	1542841	2672279
23	2107560	3650401
24	5757961	9973081
25	13623482	23596563
26	21489003	37220045
27	29354524	50843527
28	80198051	138907099
29	189750626	328657725
30	299303201	518408351
31	408855776	708158977
32	1117014753	1934726305
33	2642885282	4577611587

k	m_k	n_k
34	4168755811	7220496869
35	5694626340	9863382151
36	15558008491	26947261171
37	36810643322	63757904493
38	58063278153	100568547815
39	79315912984	137379191137
40	216695104121	375326930089
41	512706121226	888033051315
42	808717138331	1400739172541
43	1104728155436	1913445293767
44	3018173449203	5227629760075
45	7141075053842	12368704813917
46	11263976658481	19509779867759
47	15386878263120	26650854921601
48	42037733184721	72811489710961
49	99462344632562	172273834343523
50	156886956080403	271736178976085
51	214311567528244	371198523608647
52	585510091136891	1014133226193379
53	1385331749802026	2399464975995405
54	2185153408467161	3784796725797431
55	2984975067132296	5170128475599457
56	8155103542731753	14125053676996345
57	19295182152595802	33420235829592147
58	30435260762459851	52715417982187949
59	41575339372323900	72010600134783751
60	113585939507107651	196736618251755451
61	268747218386539202	465483836638294653
62	423908497265970753	734231055024833855
63	579069776145402304	1002978273411373057
64	1582048049556775361	2740187601847579969
65	3743165875258953026	6483353477106532995
66	5904283700961130691	10226519352365486021

k	m_k	n_k
67	8065401526663308356	13969685227624439047
68	22035086754287747403	38165889807614364115
69	52135575035238803162	90301464842853167277
70	82236063316189858921	142437039878091970439
71	112336551597140914680	194572614913330773601
72	306909166510471688281	531582269704753517641
73	726154884618084291242	1257737154322837808883
74	1145400602725696894203	1983892038940922100125
75	1564646320833309497164	2710046923559006391367
76	4274693244392315888531	7403985886058934882859
77	10114032809617941274226	17518018695676876157085
78	15953372374843566659921	27632051505294817431311
79	21792711940069192045616	37746084314912758705537
80	59538796254981950751153	103124220135120334842385
81	140870304450033093547922	243994524585153428390307
82	222201812645084236344691	384864829035186521938229
83	303533320840135379141460	525735133485219615486151
84	829268454325354994627611	1436335096005625752910531
85	1962070229490845368396682	3398405325496471121307213
86	3094872004656335742165753	5360475554987316489703895
87	4227673779821826115934824	7322545784478161858100577
88	11550219564299987974035401	20005567123943640205905049
89	27328112908421802064005626	47333680032365442269910675
90	43106006252543616153975851	74661792940787244333916301
91	58883899596665430243946076	101989905849209046397921927
92	160873805445874476641868003	278641604639205337129760155
93	380631510488414383527682082	659273115127619720657442237
94	600389215530954290413496161	1039904625616034104185124319
95	820146920573494197299310240	1420536136104448487712806401
96	2240683056677942685012116641	3880976897824931079610737121
97	5301513033929379567323543522	9182489931754310646934280643
98	8362343011180816449634970403	14484002965683690214257824165
99	11423172988432253331946397284	19785515999613069781581367687
100	31208688988045323113527764971	54055034964909829777420559539

Here are the first six approximations:

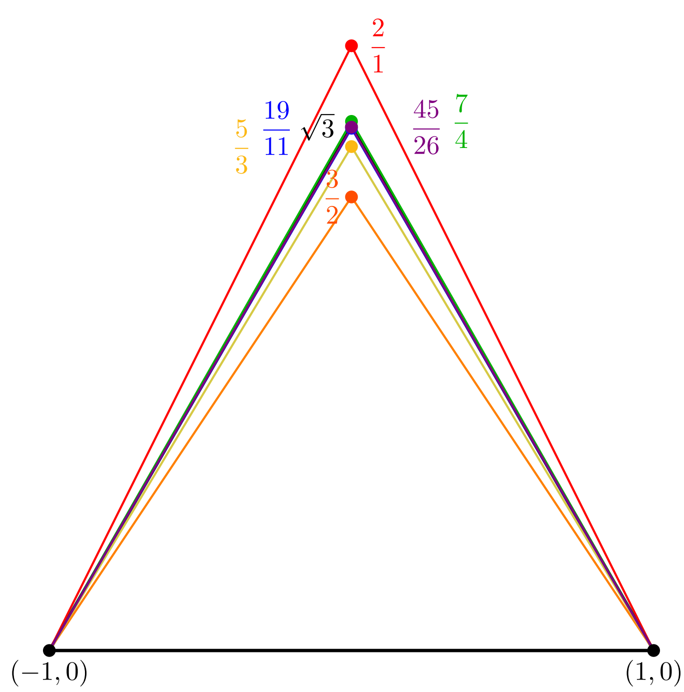

Figure 1: The first six integer-grid approximations to an equilateral triangle, normalized to have a common base.

I originally used this construction to tile grid paper with approximations of regular hexagons. In particular, I used the integer-grid hexagon with vertices

$$(-2, 0), (-1, 2), (1, 2), (2, 0), (1, -2), (-1, -2).$$

Although this hexagon is not regular, and in fact uses the worst approximation of an equilateral triangle to construct these hexagons, its vertices all lie on integer grid points, and translated copies fit together to form a hexagonal tiling. It worked, and it was much cheaper than purchasing "hex paper".

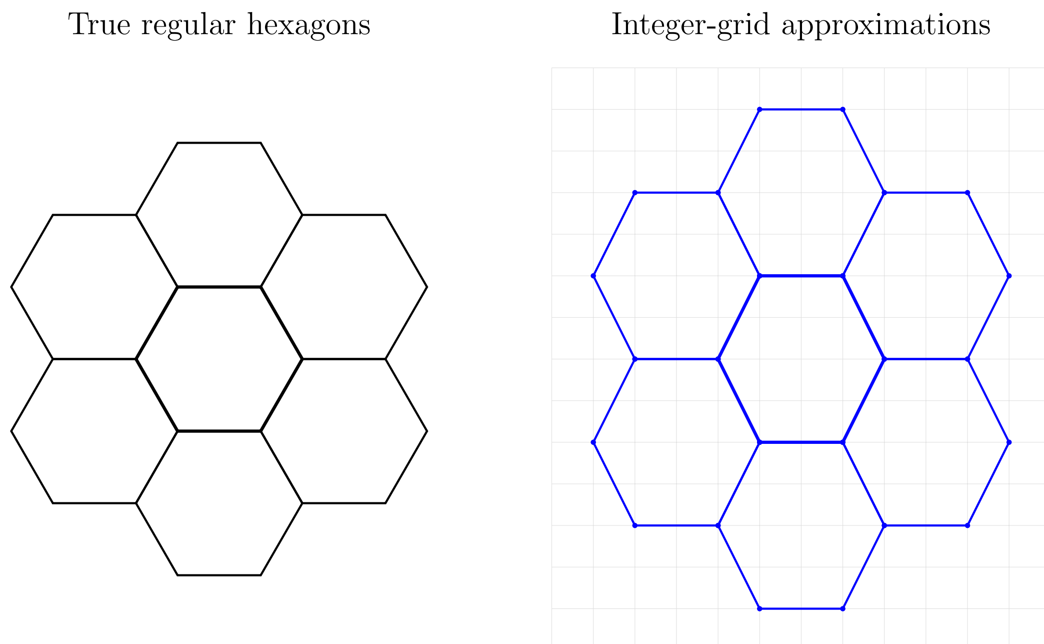

Figure 2: A cluster of seven regular hexagons compared with seven integer-grid approximations.

Memorizing 355/113

As an aside, it is easy enough to remember $\frac{22}{7}$ as an approximation to π, but given how much better $\frac{355}{113}$ is, it would be nice if there was a convenient mnemonic to remember it. Here are two:

1. The phrase "three fifty-five over one thirteen" has a pleasantly symmetrical rhythm.
2. Write the numbers $1, 1, 3, 3, 5, 5$, and then construct the number by starting with the denominator: 113 below 355. This last one was devised by Presh Talwalkar.