Secondary school knowledge for linear algebra

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Dear reader, this is a review of material that you have either already encountered in secondary school or should be able to deduce with ease from it. You are not expected to memorize every formula presented here. Rather, the goal is that, when you read through the properties, you can say to yourself: "That makes sense," or "I understand that." If you can reach that point of recognition and understanding, then you have achieved what is intended.

Contents

1 Terminology

The statement “If p, then q.” says that if p is true, then q is true. For example, “If $x > 5$, then $x > 3$.” If the first statement is true, the second statement is true, if the first statement is false, this says nothing about the second statement. As another example, “If $x \geq 2$, then $x^2 \geq 4$.” If $x \geq 2$, it is guaranteed that $x^2 \geq 4$; however, if $x < 2$, then $x^2 \geq 4$ may or may not be true.

However, suppose you wrote “If $x^2 = 4$, then $x = 2$.” This is a false statement, because $x^2 = 4$ does not guarantee that $x = 2$, as $x = -2$ is a second possibility. What you could say is “If $x^2 = 4$, then $x = 2$ or $x = -2$.” This would be a true statement.

Note that “If it is raining, then there are clouds in the sky.” is a true statement. On the other hand “If there are clouds in the sky, then it is raining.” is a false statement, for on most days, you can look up in the sky and see clouds, and yet usually it is not raining. Similarly, “If $x > 5$, then $x > 3$.” is a true statement, but the following claim would be false: “If $x > 3$, then $x > 5$.” This is because there are values of x for such as $x =$ where $4 > 3$ is true, but $4 > 5$ is clearly false. Observe also that the statement “If $x = 2$, then $x^2 = 4$.” is true; however "If $x^2 = 4$, then $x = 2$.” is false because $(-2)^2 = 4$, and yet $-2 \neq 2$.

On the other hand “p if and only if q” says that if p is true, then q is true, and if p is false, then q is also false. That is, both statements must have the same truth value. For example, “$x > 3$ if and only if $x^3 > 27$. As another example, “Given an integer n, n is even if and only if n^2 is even.”

Notice that if “p if and only if q” is a true statement, then both “if p, then q” and “if q, then p” must also be true statements. For example, from the above, we note that “If $x > 3$, then $x^3 > 27$.” is true, and so is “If $x^3 > 27$, then $x > 3$.”

For example, “I see lightning if and only if I hear thunder.” is a false statement, because it is possible to see lightning on the horizon, and yet, the distance is so far that by the time the sound gets to you, it is no longer audible. However, this statement is true: “If lightning strikes within one kilometer, then I see lightning if and only if I hear thunder.” is a true statement, for if lightning strike is that close, you will most certainly both see it and hear it (and, incidentally, feel it).

2 Equality

Here are the properties of equality:

1. Everything is equal to itself, so $x = x$.
2. $x = y$ if and only if $y = x$.
3. If $x = y$ and $y = z$, then $x = z$.

Throughout this review, we will see many other examples of properties related to equality.

3 Negation

The following are true if x and y are both integers, both rational numbers, or both real numbers.

1. $-x$ changes the sign of the number x.
2. $-x$ is that value such that $x + (-x) = 0$. E.g, $5 + (-5) = 0$.
3. $x > 0$ if and only if $-x < 0$, and also $x \geq 0$ if and only if $-x \leq 0$.
4. $-0 = 0$.
5. $-x = x$ if and only if $x = 0$.
6. $-(-x) = x$.
7. $x < y$ if and only if $-x > -y$, and also $x \leq y$ if and only if $-x \geq -y$.

Some terminology:

1. The *positive* integers, rational numbers, and real numbers are all those numbers that satisfy $x > 0$.
2. The *negative* integers, rational numbers, and real numbers are all those numbers that satisfy $x < 0$.
3. 0 is neither positive nor negative.
4. x is positive if and only if $-x$ is negative.
5. The *non-negative* integers, rational numbers, and real numbers are all those numbers that satisfy $x \geq 0$.
6. The *non-positive* integers, rational numbers, and real numbers are all those numbers that satisfy $x \leq 0$.
7. 0 is both non-negative and non-positive.
8. x is non-positive if and only if $-x$ is non-negative.

4 Collections of numbers

1. The collection of all integers is sometimes written as **Z**, from the German word for the integers: *Zahlen*.

2. The collection of all rational numbers is sometimes written as **Q**, from the word "quotients," meaning all ratios of two integers with the denominator being non-zero.

3. The collection of all real numbers is sometimes written as **R**.

Every integer n is equal to the rational number $\frac{n}{1}$ and equal to the real number as a decimal representation $n.000\cdots$. Every rational number can be represented by a real number with a terminating or repeating decimal representation (although, terminating is technically a repeating 0). Every rational number can be written such that the denominator is a positive integer and the greatest common factor of the numerator and denominator is 1. For example, $\frac{-1}{-2}$, $\frac{5}{10}$ and $\frac{1}{2}$ are all equal, except the last is the preferred representation. Similarly, $\frac{-70}{42}$, $\frac{5}{-3}$ and $\frac{-5}{3}$ are also all equal, except the last is the preferred representation. To find the decimal representation of a rational number, perform long division until you reach a stage where you are bringing down zeros after the decimal point and the remainder begins to repeat.

To convert a number in the decimal representation with a repeating decimal expansion to a rational number, this is more easily shown with an example: if $q = 3.174\overline{15}$, then the repetition is two digits, so multiply by $10^2 = 100$, so $100q = 317.4\overline{15}$. Next, subtract the first equation from the second to get $99q = 314.241$. To make the right-hand side an integer, multiply both sides by 1000, so $99000q = 314241$. Dividing both sides by 99000 yields $\frac{314241}{99000}$, and dividing out the greatest common factor of 3 yields $\frac{104747}{33000}$.

Every real number that does not have a terminating or repeating decimal representation is an "irrational number".

Remark

> It is important that you understand you don't have to know how to do these conversions. As an engineer, you must understand that the conversions can be done, and that if you try hard enough, you can understand that conversion.

Thus,

1. All integers are rational numbers.

2. All rational numbers are real numbers.

3. Every real number is either a rational number or an irrational number.

Two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ are equal if $ad = bc$. If $b > 0$ and $d > 0$, then $\frac{a}{b} < \frac{c}{d}$ if and only if $ad < bc$. If one of the denominators is negative, then $\frac{a}{b} < \frac{c}{d}$ if and only if $ad > bc$; however, given the preferred representation has a denominator that is a positive integer, this should be the exception and not the rule: $\frac{1}{-3} < \frac{1}{2}$ because $2 > -3$ while $\frac{-1}{3} < \frac{1}{2}$ because $-2 < 3$.

For real numbers written in decimal form, the representation is unique in almost all cases. The one exception occurs when the decimal expansion ends with repeating 9s. In such cases, the repeating 9s can be replaced by 0s, and the digit immediately to the left is increased by 1. This is the preferred decimal representation. Thus,

$$99.\overline{9} = 100, \quad 49.\overline{9} = 50, \quad 45.\overline{9} = 46, \quad 3.15\overline{9} = 3.16.$$

Why does this happen? Consider the repeating decimal

$$n = 0.\overline{9} = 0.999999\cdots.$$

Therefore, $10n = 9.\overline{9} = 9.999999\cdots$. Therefore $10n - n = 9n = 9.0$, and as $9n = 9$, it also follows that $n = 1$. You can also see that this is an infinite geometric sum:

$$\sum_{k=0}^{\infty} 0.9 \cdot 0.1^k = 0.9 + 0.09 + 0.009 + 0.0009 + \cdots.$$

The first value is 0.9 and the common ratio is 0.1, so the sum is

$$0.9 \cdot \frac{1}{1 - 0.1} = \frac{0.9}{0.9} = 1.$$

Therefore,

$$0.\overline{9} = 1.$$

If you forgot what a geometric sum is, there is a section on that topic below.

5 Absolute value

The absolute value is defined as follows: $|x| = x$ if $x \geq 0$, and $|x| = -x$ if $x < 0$, and this is true for all integers, rational numbers, and real numbers. This is sometimes written as $|x| = \left\{ \begin{array}{rl} x & x \geq 0 \\ -x & x < 0 \end{array} \right.$.

The properties include:

1. $|x| \geq 0$.
2. $|x| = 0$ if and only if $x = 0$.
3. $|x| = |-x|$.

6 Addition

The following are true if x, y and z are all integers, all rational numbers, or all real numbers.

1. If $x + y = z$, then x and y are called "terms" and z is the "sum".
2. If you add two integers, the sum is an integer; if you add two rational numbers, the sum is a rational number; and if you add two real numbers, the sum is a real number.
3. $x + y = y + x$.
4. $x + 0 = 0 + x = x$.
5. $x + y = x$ if and only if $y = 0$.
6. $-x$ is the unique number such that $x + (-x) = (-x) + x = 0$.
7. $x + y = 0$ if and only if $y = -x$ and $x = -y$.
8. Given n numbers, it doesn't matter what order you add them in.
9. Specifically, $(x + y) + z = x + (y + z)$, so given three numbers, it matters not if you begin by adding the first two and then adding the third, or if you begin by adding the latter two and then adding to this the first.
10. $x + y = x + z$ if and only if $y = z$.
11. $x + y \leq x + z$ if and only if $y \leq z$.
12. $x + y < x + z$ if and only if $y < z$.
13. $-(x + y) = (-x) + (-y)$.
14. $|x + y| \leq |x| + |y|$. Moreover, $|x + y| < |x| + |y|$ if and only if one is positive and the other is negative. For example, $|3 + (-4)| = 1$ but $|3| + |-4| = 7$.

7 Subtraction

The following are true if x, y and z are all integers, all rational numbers, or all real numbers.

1. If $x - y = z$, then x and y are called "terms" and z is the "difference".
2. If you subtract two integers, the difference is an integer; if you subtract two rational numbers, the difference is a rational number; and if you subtract two real numbers, the difference is a real number.
3. $x - y = x + (-y)$.
4. $x - y = -(y - x)$.
5. $x - y = y - x$ if and only if $x = y$, so $x - y = y - x = 0$.
6. $x - 0 = x$.
7. $0 - x = -x$.
8. $x - x = 0$.
9. $x - y = 0$ if and only if $x = y$.
10. $x - y = x$ if and only if $y = 0$.
11. $x - z = y - z$ if and only if $x = y$.
12. $z - x = z - y$ if and only if $x = y$.
13. $(x - y) - z = x - (y - z)$ if and only if $z = 0$. If you see $x - y - z$, you must assume that you are to perform the operations from left to right (so, $(x - y) - z$).
14. $x - z \leq y - z$ if and only if $x \leq y$.
15. $x - y \leq 0$ if and only if $y - x \geq 0$.
16. $x - z < y - z$ if and only if $x < y$.
17. $x - y < 0$ if and only if $y - x > 0$.
18. $|x - y| = |y - x|$.

8 Multiplication

The multiplication of two numbers x and y may be written as xy, $x \cdot y$ or $x \times y$, although the third is discouraged.

The following are true if x, y and z are all integers, all rational numbers, or all real numbers.

1. If $xy = z$, then x and y are called "terms" and z is the "product".
2. If you multiply two integers, the product is an integer; if you multiply two rational numbers, the product is a rational number; and if you multiply two real numbers, the product is a real number.
3. $xy = yx$.
4. $x \cdot 1 = 1 \cdot x = x$.
5. $x \cdot 0 = 0 \cdot x = 0$.
6. $-x = (-1) \cdot x$.
7. $xy = 0$ if and only if $x = 0$ or $y = 0$.
8. $xy = x$ if and only if $x = 0$ or $y = 1$.
9. If $x \neq 0$, $xy = x$ if and only if $y = 1$.
10. If $z \neq 0$, then $xz = yz$ if and only if $x = y$.
11. If $x \neq y$, then $xz = yz$ if and only if $z = 0$.
12. $x(y + z) = xy + xz$, or we say that multiplication *distributes* over addition.
13. $-(xy) = (-x)y = x(-y)$.
14. $xy = (-x)(-y)$.
15. Given n numbers, it doesn't matter what order you multiply them in.
16. Specifically, $(xy)z = x(yz)$, so given three numbers, it matters not if you begin by multiplying the first two and then multiplying the third, or if you begin by multiplying the latter two and then multiplying to this the first.
17. $xz = yz$ if and only if $z = 0$ or $x = y$.
18. If $x > 0$, then $y < z$ if and only if $xy < xz$.
19. If $x < 0$, then $y < z$ if and only if $xy > xz$.
20. $xy > 0$ if and only if both x and y have the same sign.
21. $xy < 0$ if and only if both x and y have opposite signs.

22. $|xy| = |x||y|$.

23. $|xy| = 0$ if and only if $x = 0$, $y = 0$ or both.

24. If $x \neq 0$, $|xy| = |x|$ if and only if $y = \pm 1$.

9 Reciprocals

We now diverge and focus only on rational numbers and real numbers. The concept of a reciprocal does not really apply to integers. Given a non-zero number x, its reciprocal (if it has one), is that number we denote as x^{-1} so that $x \cdot x^{-1} = 1$.

For the integers, only two integers have reciprocals: 1 and -1, and $1^{-1} = 1$ and $(-1)^{-1} = -1$.

Theorem 1. No integers other than 1 and -1 have reciprocals.

Proof. We want to show that the only integers that have reciprocals (that is, another integer you can multiply them by to get 1) are 1 and -1. We first observe that if n is a non-zero integer, then $|n| \geq 1$.

1. **Step 1: What we are looking for**
   Suppose n is an integer. We want to know if there is another integer m such that
   $$n \cdot m = 1.$$

2. **Step 2: Think about the size of the numbers**
   If $|n| \geq 2$, then multiplying n by any non-zero integer m makes the number at least 2 in size:
   $$|n \cdot m| = |n| \cdot |m| \geq 2 \cdot 1 = 2.$$

3. **Step 3: Why this causes a problem**
   If $|n \cdot m| \geq 2$, then $n \cdot m$ can never equal 1, because 1 is smaller than 2.

4. **Step 4: The only possibilities left**
   That means the only integers n that might work are those with $|n| = 1$. These are:
   $$n = 1 \quad \text{or} \quad n = -1.$$

5. **Step 5: Check them**

   If $n = 1$, then $m = 1$ works, because $1 \cdot 1 = 1$.

   If $n = -1$, then $m = -1$ works, because $(-1) \cdot (-1) = 1$.

6. **Final Answer**
   Therefore, the only integers that have reciprocals among the integers are 1 and -1.

$\square$

Traditionally, proofs end with a square symbol.

Given a non-zero rational number q, then it can be uniquely written as $q = \frac{m}{n}$ where m and n are integers, $m \neq 0$, $n \geq 0$, and $\gcd(|m|, n) = 1$. In this case, the reciprocal of q is $q^{-1} = \frac{n}{m}$. This is true because $\frac{m}{n} \cdot \frac{n}{m} = \frac{mn}{nm} = 1$.

It is more difficult to calculate the reciprocal of a real number, at least when it has an infinite non-repeating decimal representation. If the decimal representation is repeating or terminating, it is easiest to convert the number into a rational number, calculate the reciprocal, and convert that back into the decimal representation.

The following are true if x and y are both non-zero rational numbers or both non-zero real numbers.

1. $(x^{-1})^{-1} = x$.
2. $xy = 1$ if and only if $y = x^{-1}$ and $x = y^{-1}$.
3. $(-x)^{-1} = -x^{-1}$.
4. $(xy)^{-1} = x^{-1}y^{-1} = y^{-1}x^{-1}$.
5. $|x^{-1}| = |x|^{-1}$.
6. $|xx^{-1}| = |1| = 1$, so
   (a) $|x| > 1$ if and only if $0 < |x^{-1}| < 1$.
   (b) $|x| = 1$ if and only if $|x^{-1}| = 1$.
   (c) $0 < |x| < 1$ if and only if $|x^{-1}| > 1$.
7. $x > 0$ if and only if $x^{-1} > 0$.
8. $x < 0$ if and only if $x^{-1} < 0$.
9. $0 < x < y$ if and only if $0 < y^{-1} < x^{-1}$.
10. $y < x < 0$ if and only if $x^{-1} < y^{-1} < 0$.
11. $x^{-1} + y^{-1} = (xy)^{-1}(x + y)$.
12. $x = x^{-1}$ if and only if $x = \pm 1$.
13. If $x > 0$, $x + x^{-1} \geq 2$.
14. If $x < 0$, $x + x^{-1} \leq -2$.

10 Division

The division of two numbers x and y may be written as x/y, $\frac{x}{y}$ or $x \div y$, although the third is not commonly used outside of elementary and secondary schools. Division is never defined if $y = 0$, so any time we write x/y, we implicitly assume that $y \neq 0$.

If x and y are integers, then x/y is an integer if and only if $\gcd(x, y) = |y|$; that is, it is only defined if the numerator is an integer multiple of the denominator. Put another way, x/y is an integer if and only if $x = zy$ for some integer z. For example, $3/4$ is not an integer, neither is $3/42$, but $546/42 = 13$. Given two integers, what we can say is the following: if x and y are integers, then there are unique integers s and r with $0 \leq r < |y|$ such that $x = sy + r$. For example, $17 = 3 \times 5 + 2$. We call s the "quotient" and r the "remainder".

The following are true if x, y and z are all rational numbers, or all real numbers.

1. If $x/y = z$, then x is the "dividend", y is the "divisor", and z is the "ratio" or "quotient". If you divide two rational numbers, the ratio is a rational number, and if you divide two real numbers, the ratio is a real number.
2. If $y \neq 0$, then $x/y = xy^{-1}$.
3. If $x \neq 0$ and $y \neq 0$, then $(x/y)^{-1} = y/x = yx^{-1}$.
4. If $x \neq 0$ and $y \neq 0$, then $(x/y) = (y/x)^{-1} = xy^{-1}$.
5. $x/1 = x$.
6. $\frac{1}{x} = 1/x = x^{-1}$.
7. $x/-1 = -x$.
8. $x/y = 1$ if and only if $x = y$.
9. $x/y = x$ if and only if $x = 0$ or $y = 1$.
10. $x/z = y/z$ if and only if $x = y$.
11. $z/x = z/y$ if and only if $z = 0$ or $x = y$.
12. If $x \neq 0$, $y \neq 0$, and $x \neq -y$, then $\frac{1}{x^{-1}+y^{-1}} = \frac{xy}{x+y}$
13. If $x \neq 0$ and $z \neq \pm 1$, then $(x/y)/z \neq x/(y/z)$. If you see $x/y/z$, you must assume that you are to perform the operations from left to right (so, $(x/y)/z$).
14. $x/z \leq y/z$ if and only if both $x \leq y$ and $z > 0$ or both $x \geq y$ and $z < 0$.
15. $0 < x/y < 1$ if and only if $y/x > 1$.
16. $-1 < x/y < 0$ if and only if $y/x < -1$.
17. $|x/y| = |x|/|y| = |x||y|^{-1}$.

11 Exponents

For integers, rational numbers, and real numbers, we may define x^n for a non-negative n as follows: If $n = 0$, then $x^0 = 1$, otherwise, $x^n = \underbrace{x \cdots x}_{n \text{ times}}$. While technically, 0^0 is indeterminate, it is a matter of convenience to define $0^0 = 1$.

Because the reciprocal is not defined for integers (other than ± 1), for rational numbers and real numbers, we may define x^n for $n < 0$ as $(x^{-1})^{-n}$. So $2.5^{-3} = (2.5^{-1})^3 = 0.4^3 = 0.064$. Note that 0^n is not defined if $n < 0$.

If the exponent a is a rational number, to calculate x^a:

1. When the base is zero, so $x = 0$:
   (a) If $a = 0$, we define $0^0 = 1$.
   (b) If $a > 0$, then $0^a = 0$.
   (c) If $a < 0$, 0^a is undefined.
2. When the base is positive, so $x > 0$:
   (a) If $a = 0$, $x^0 = 1$.
   (b) If $a = \frac{m}{n} > 0$, we define $x^a = (\sqrt[n]{x})^m$.
   (c) If $a = -\frac{m}{n} < 0$, we find $(x^{-1})^{-a} = (\sqrt[n]{x^{-1}})^m$.

Remark

Defining powers when the exponent is real or when the base is negative (so $x < 0$) is best left for when you understand complex numbers.

For example, $0.16^{\frac{3}{2}} = \sqrt{0.16}^3 = 0.4^3 = 0.064$, while $0.16^{-\frac{3}{2}} = \sqrt{0.16^{-1}}^3 = \sqrt{6.25}^3 = 2.5^3 = 15.625$.

The following properties hold for all integer exponents and all possible non-zero bases:

1. $(x^a)^b = x^{ab}$ requires $x \neq 0$.

The following properties hold for all exponents (integers, rational numbers, or real numbers) and all non-negative real bases x (and for $x \neq 0$ where needed):

1. $x^a x^b = x^{a+b}$.
2. $x^a = (x^{-a})^{-1}$ when $x \neq 0$.
3. If $a < b$ and $x > 1$, then $x^a < x^b$.
4. If $a < b$ and $0 < x < 1$, then $x^a > x^b$.

The following properties also hold for all exponents, but require all values in the base to be positive:

1. $(xy)^a = x^a y^a$ requires $x > 0$ and $y > 0$.
2. If a and b are not integers, then $(x^a)^b = x^{ab}$ requires $x \geq 0$.

For square roots and more generally for even n-th roots with n a positive even integer:

1. $\sqrt{xy} = \sqrt{x}\sqrt{y}$ requires $x \geq 0$ and $y \geq 0$.
2. $\sqrt{x^2} = |x|$ for all x.
3. For n being a positive even integer, $\sqrt[n]{xy} = \sqrt[n]{x}\,\sqrt[n]{y}$ requires $x \geq 0$ and $y \geq 0$.

12 Polynomials

You should be aware of the following properties of polynomials:

1. Given a non-negative integer n and $n+1$ numbers $a_0, a_1, \ldots, a_n$ that are all real numbers where $a_n \neq 0$, for a polynomial p defined as $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$:
   (a) The degree of the polynomial is n.
   (b) We will say that $\deg(p) = n$.
   (c) Each of $a_n x^n$, $a_{n-1}x^{n-1}, \ldots, a_2x^2, a_1x$ and a_0 are called "terms".
   (d) Each of the real numbers $a_0, \ldots, a_n$ are called "coefficients".
   (e) If the degree of the polynomial is n, then we say that all higher coefficients are zero, so $a_{n+1} = a_{n+2} = \cdots = 0$.
   (f) a_n is specifically described as the "leading coefficient".
   (g) a_0 is specifically described as the "constant coefficient" or the "constant term".
   (h) If $a_n = 1$, it is called a "monic polynomial".
   (i) If $a_{n-1} = a_{n-2} = \cdots = a_2 = a_1 = a_0 = 0$ (so only the leading coefficient is non-zero), it is called a "monomial".
2. If the degree of a polynomial is 0, we call it a constant polynomial.
3. There is a special polynomial when all the coefficients are zero. This is called the "zero polynomial", and is written as *0*, so *0*$(x) = 0$ for all x. To avoid confusion, any time we use *0*, we will remind you that this is the zero polynomial. The degree of the zero polynomial is not well defined: it is mathematically convenient to define it as $-\infty$, but in many programming languages, it is given as -1.
4. If two polynomials p and q have different degrees, they are not equal, so $p \neq q$.
5. If $p(x) = a_n x^n + \cdots + a_0$ and $q(x) = b_n x^n + \cdots + b_0$, then $p = q$ if and only if $a_n = b_n, \ldots, a_0 = b_0$.
6. Given a polynomial p, if there is a point r such that $p(r) = 0$, we call r a "root" of the polynomial p.
7. A non-constant polynomial of degree n has at most n roots.
8. If a polynomial is written as $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$, we will say that it is written in "expanded form".
9. If a polynomial has n roots and is written as $a_n(x - r_1)(x - r_2)\cdots(x - r_n)$ where $r_1, \ldots, r_n$ are the n roots, we will say that it is written in "factored form".
10. To multiply a polynomial by a real number c is to multiply each term by that scalar, so if $p(x) = a_n x^n + \cdots + a_1x + a_0$, then cp is the polynomial where $(cp)(x) = (ca_n)x^n + \cdots + ca_1x + ca_0$. This is known as "scalar multiplication".

11. Multiplying a polynomial by a non-zero real number does not change the roots.

12. Multiplying a polynomial by 1 leaves the polynomial unchanged, so $1 \cdot p = p$.

13. If the only coefficients of a polynomial that are non-zero are when the degree of the term is even, we call the polynomial "even" and if p is such a polynomial, then $p(-x) = p(x)$. For example, $8.1x^4 - 3.7x^2 + 5.9$ is even.

14. If the only coefficients of a polynomial that are non-zero are when the degree of the term is odd, we call the polynomial "odd" and if p is such a polynomial, then $p(-x) = -p(x)$. For example, $-4.2x^7 - 6.7x^5 + 5.9x^3 + 5.8x$ is odd.

15. The zero polynomial is the only polynomial that is both even and odd.

12.1 Summation notation

In this section, we will generally write polynomials out, but we will also introduce a compact notation called "summation" notation. It uses the Greek capital letter sigma, Σ. The notation

$$\sum_{k=m}^{n} \mathit{term}$$

where m and n are integers and where $m \leq n$ means that we will let k take on each integral value between those two endpoints, inclusive, and evaluate the term with k taking on that value. The integer m is called the "lower limit", the integer n is called the "upper limit", k is the "index of summation", and each evaluated expression is called a "term". We will then sum all of those terms.

$$\sum_{k=1}^{5} k = 1 + 2 + 3 + 4 + 5,$$

while

$$\sum_{k=5}^{10} k^2 = 25 + 36 + 49 + 64 + 81 + 100.$$

The term may not, however, immediately evaluate to a number, such as

$$\sum_{k=-2}^{2} a_k$$

is equivalent to $a_{-2} + a_{-1} + a_0 + a_1 + a_2$. It is assumed that these variables do have actual values, but they may be unknown, or this may express a formula. Summation notation is especially useful for formulas. For example, the average (mean) of 10 values $a_1, \ldots, a_{10}$ is

$$\overline{a} = \frac{1}{10} \sum_{k=1}^{10} a_k,$$

and we use this average to define the standard deviation as follows:

$$\sigma = \sqrt{\frac{1}{10}\sum_{k=1}^{10}(a_k - \overline{a})^2}.$$

We may not even know how many items we are averaging, so we may use an unknown n as our upper limit with

$$\overline{a} = \frac{1}{n}\sum_{k=1}^{n} a_k$$

and

$$\sigma = \sqrt{\frac{1}{n}\sum_{k=1}^{n}(a_k - \overline{a})^2}.$$

If we had $n = 20$ values, we'd sum those twenty values, if $n = 100$, then we would sum those one hundred values.

Given a sum with an upper limit n, we may also consider the same sum but only up to a smaller index $m < n$. This is called a "partial sum" of the full sum. For example, if all the terms are non-negative, then every partial sum is less than or equal to the total sum. That is, for $m \leq n$,

$$\sum_{k=0}^{m} |a_k| \leq \sum_{k=0}^{n} |a_k|.$$

We may even calculate the sum of an infinite number of terms. For example, consider the infinite sum often associated with Zeno's paradox:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \cdots .$$

This can be written as

$$\sum_{k=1}^{\infty} \frac{1}{2^k}.$$

To understand its value (if it has one), we look at the partial sums

$$S_n = \sum_{k=1}^{n} \frac{1}{2^k}.$$

In this case, it can be shown that

$$S_n = 1 - \frac{1}{2^n}.$$

This sequence of partial sums, or "series", is

$$\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}, \frac{63}{64}, \ldots,$$

and as n grows larger and larger, S_n approaches 1. We therefore say that this series "converges" to 1, and we assign the infinite sum this value.

By contrast,

$$\sum_{k=1}^{n} 1 = n,$$

and as n increases without bound, this sum also increases without bound. Hence the infinite sum

$$\sum_{k=1}^{\infty} 1$$

is said to "diverge".

One way to compactly write a polynomial is to use summation notation: the polynomial p evaluated at x is the value

$$p(x) = \sum_{k=0}^{n} a_k x^k,$$

where $x^0 = 1$ and the degree of p is assumed to be n, so $a_n \neq 0$. A constant polynomial q with the value a_0 simplifies to

$$q(x) = \sum_{k=0}^{0} a_k x^k = a_0.$$

Remark

When we add discrete values, such as terms in a polynomial, we use the summation notation. You will see in calculus, however, that integrals are infinite sums, and they use that curly "S" shape $\int$. The reason it looks like an "S" is because it was the German and old Engliſh "s", ſo long as that letter was not the laſt letter in the word.

12.2 Polynomial addition and subtraction

These are the properties of polynomial addition where $p(x) = a_n x^n + \cdots + a_0$ and $q(x) = b_m x^m + \cdots + b_0$:

1. We define $p + q$ to be that polynomial that has coefficients $a_k + b_k$ for all k, recalling that $a_k = 0$ for $k > n$ and $b_k = 0$ for all $k > m$.

2. If $m \neq n$, then the degree of $p + q$ is $\max\{m, n\}$.

3. If $m = n$, then the degree of $p + q$ is less than or equal to that common degree, where we may have a strict inequality if there is a cancellation of terms, so $a_n = -b_n$, etc.

4. If the coefficients of q are the coefficients of p multiplied by -1, so $a_n = -b_n$, $a_{n-1} = -b_{n-1}$, $\ldots$, $a_1 = -b_1$ and $a_0 = -b_0$, then $p + q = \mathit{0}$, the zero polynomial. This polynomial often written as $-p$, so $p + (-p) = \mathit{0}$. For example, if we add $4x^3 - 2x + 7$ to the polynomial $-4x^3 + 2x - 7$, we get the zero polynomial.

5. If we multiply p by the real number -1, we get $-p$, so $-p = (-1)p$. For example, given $3x^2 - 4$, multiplying this by -1 gives us $-3x^2 + 4$, and adding these together gives us the zero polynomial.

You will note that many of the properties of polynomial addition are shared by the properties of addition and subtraction of rational numbers and real numbers, at least, those that do not relate to inequalities. Given polynomials p, q and r:

1. p and q, $p + q = q + p$.
2. $p + \mathit{0} = p$.
3. $p + q = p$ if and only if $q = \mathit{0}$, the zero polynomial.
4. $-p$ is the unique polynomial such that $p + (-p) = (-p) + p = \mathit{0}$, the zero polynomial.
5. $p + q = \mathit{0}$ if and only if $p = -q$ and $q = -p$.
6. Given N polynomials, it doesn't matter what order you add them in.
7. Specifically, $(p + q) + s = p + (q + s)$, so given three polynomials, it matters not if you begin by adding the first two and then adding the third, or if you begin by adding the latter two and then adding to this the first.
8. $p + q = p + s$ if and only if $q = s$.
9. $-(p + q) = -p + -q$.
10. $p - q = p + (-q)$.
11. $p - q = -(q - p)$.
12. $p - q = q - p$ if and only if $p = q$, so $p - q = q - p = \mathit{0}$, the zero polynomial.
13. $p - \mathit{0} = p$.
14. $\mathit{0} - p = -p$.
15. $p - p = \mathit{0}$.
16. $p - q = \mathit{0}$ if and only if $p = q$.
17. $s - p = s - q$ if and only if $p = q$.
18. If $s \neq \mathit{0}$ (the zero polynomial), then $(p - q) - s \neq p - (q - s)$. If you see $p - q - s$, you must assume that you are to perform the operations from left to right (so $(p - q) - s$).

Remark

This is one of the most important observations in mathematics: that different *objects* can never-the-less have similar properties under corresponding definitions of addition. Thus, if you understand the properties in one context, you should also expect similar properties in the other.

12.3 Polynomial multiplication

For polynomial multiplication, the product of two terms $a_j x^j$ and $b_k x^k$ is defined as $(a_j b_k)x^{j+k}$, and the product of two polynomials is where you multiply each term of one polynomial by each term of the other, and then add them together. Generally speaking, this is more computationally expensive, and we would generally leave this to a computer to perform such calculations, but you should be aware of reasonably straight-forward products of lower-degree polynomials:

$$(ax + b)(cx + d) = (ac)x^2 + (ad + bc)x + (bd).$$

You remembered this using the term FOIL: first, outside, inside, and last.

Note that FOIL is just distributivity: $(ax + b)(cx + d) = cx(ax + b) + d(ax + b)$ using the distributive law

1. $pq = qp$.
2. $p\mathit{0} = \mathit{0}p = \mathit{0}$.
3. If either p or q is the zero polynomial, the product pq has the degree equal to the zero polynomial.
4. Otherwise, $\deg(pq) = \deg(p) + \deg(q)$ because $a_n x^n \cdot b_m x^m = (a_n b_n)x^{m+n}$.
5. $pq = \mathit{0}$ if and only if $p = \mathit{0}$ or $q = \mathit{0}$.
6. $pq = p$ if and only if q is the constant polynomial $q(x) = 1$.
7. If $s \neq \mathit{0}$, then $ps = qs$ if and only if $p = q$.
8. If $p \neq q$, then $ps = qs$ if and only if $s = \mathit{0}$.
9. $p(q+s) = pq+ps$, or we say that polynomial multiplication *distributes* over polynomial addition.
10. $-(pq) = (-p)q = p(-q)$.
11. $pq = (-p)(-q)$.
12. Given N polynomials, it doesn't matter what order you multiply them in.
13. Specifically, $(pq)s = p(qs)$, so given three polynomials, it matters not if you begin by multiplying the first two and then multiplying by the third, or if you begin by multiplying the latter two and then multiply this by the first.

14. $ps = qs$ if and only if $s = 0$, $p = q$, or both.

Remark

Once again, hopefully you see many of the same parallels between the multiplication of rational numbers and real numbers, and the multiplication of polynomials.

12.4 Polynomial division

At this point, the relationship between polynomial division and the division of rational numbers and real numbers, because we can define polynomial division p/q, but the ratio is a polynomial if and only if all the roots of the divisor q are also roots of the dividend p. Similarly, p/q is defined if and only if there is a polynomial s such that $p = sq$. You will note, however, that this is very similar to integer division.

If p and q are polynomials, then p/q is a polynomial if and only if all the roots of q are also roots of p counting multiplicity; that is, it is only defined if the numerator is the denominator multiplied by another polynomial. For example, $(x-3)/(5x+4)$ is not an polynomial, and neither is $(3x-4)/(4x^2+2x+1)$, but $(24x^3-20x^2-10x-8)/(4x^2+2x+1) = 6x-8$. As another example, $2x^2+6x+4$ is divisible by $4x+4$, and the ratio is $0.5x+1$, but that same quadratic is not divisible by $x-1$, nor is $2x^2-6x+4$ divisible by $4x+4$.

Given two polynomials, what we can say is the following: if p and q are polynomials, then we can always write $p = sq + r$ where s and r are polynomials and the degree of r is less than the degree of q. We call s the "quotient" and r the "remainder".

Looking at all the previous examples, we see that:

$$x - 3 = 0.2(5x+4) - \tfrac{19}{5}$$

$$3x - 4 = 0 \cdot (4x^2+2x+1) + (3x-4)$$

$$24x^3 - 20x^2 - 10x - 8 = (6x-8)(4x^2+2x+1) + 0 \text{ (zero remainder)}$$

$$2x^2 + 6x + 4 = (0.5x+1)(4x+4) + 0 \text{ (again, zero remainder)}$$

$$2x^2 + 6x + 4 = (2x+8)(x-1) + 12$$

$$2x^2 - 6x + 4 = (0.5x-2)(4x+4) + 12$$

Thus, the properties of polynomial division parallel more closely the properties of integer division.

However, you did learn a few properties about polynomial division in secondary school:

1. If you divide a polynomial $p(x)$ by the monic polynomial $x - r$, the remainder will be equal to $p(r)$, the original polynomial evaluated at $x = r$.
2. If $p(r) = 0$, then r is a root of that polynomial, and when you divide $p(x)$ by $x - r$, the result will be $p(x) = (x-r)q(x) + 0$.

12.5 Quadratic polynomials

There are very specific properties of a quadratic polynomial you learned. We will write the polynomial $p(x) = ax^2 + bx + c$ where $a \neq 0$.

1. A quadratic polynomial is often referred to as simply a "quadratic", with the polynomial being assumed.
2. The maximum or minimum of the quadratic is at the point $x = -\frac{b}{2a}$.
3. If $a > 0$, the polynomial grows *upward* from the minimum, but we will say that it is "concave up".
4. If $a < 0$, the polynomial grows *downward* from the maximum, but we will say that it is "concave down".
5. The y-intercept is the point $(0, c)$.
6. The "discriminant" of the polynomial is $b^2 - 4ac$. It allows you to distinguish, or *discriminate*, between the two roots.
7. If $b^2 - 4ac < 0$, the polynomial has no real roots.
8. If $b^2 - 4ac = 0$, the polynomial has a "double root" at $x = -\frac{b}{2a}$.
9. If $b^2 - 4ac > 0$, the polynomial has two unique roots centered around $x = -\frac{b}{2a}$: $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$, which can be written as $-\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$.
10. The x-intercept are the roots, if any.
11. If you wrote the polynomial in the form $a\left(x + \frac{b}{2a}\right)^2 + p\left(-\frac{b}{2a}\right)$, you called this the "vertex form".
12. The quadratic p is "symmetric" around $x = -\frac{b}{2a}$, meaning that $p\left(-\frac{b}{2a} + x\right) = p\left(-\frac{b}{2a} - x\right)$ for any real number x. $x = -\frac{b}{2a}$ is described as the "axis of symmetry".

Observe that if we add the two roots, we get

$$\left(-\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right) + \left(-\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right) = -\frac{b}{2a} - \frac{b}{2a} = -\frac{b}{a}.$$

Observe that if we multiply the two roots, we get

$$\left(-\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right)\left(-\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right) = \frac{b^2}{4a^2} - \frac{b^2 - 4ac}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}.$$

12.6 Binomial coefficients

If n is a non-negative integer, then there are easier ways to calculate $(x+y)^n$ and $(x+1)^n$ than multiplying $x+y$ or $x+1$ by itself n times. Instead, we define the binomial coefficient $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. This is the number of ways that $k = 0, \ldots, n$ objects may be chosen from n distinct objects. Thus, we have the following expansions:

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-2}x^2y^{n-2} + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n.$$

Observing that from n unique objects, there is only one way of choosing none of them or all of them, and given those objects, there is n ways of choosing one of them or all but one of them, thus we have that $\binom{n}{0} = \binom{n}{n} = 1$ and $\binom{n}{1} = \binom{n}{n-1} = n$, we can simplify this to:

$$(x+y)^n = x^n + nx^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-2}x^2y^{n-2} + nxy^{n-1} + y^n.$$

If we replace this with $y = 1$, we get that

$$(x+1)^n = x^n + nx^{n-1} + \binom{n}{2}x^{n-2} + \cdots + \binom{n}{n-2}x^2 + nx + 1.$$

You may recall that binomial coefficients follow a pattern:

	k	0	1	2	3	4	5	6	7	8
n										
0		1								
1		1	1							
2		1	2	1						
3		1	3	3	1					
4		1	4	6	4	1				
5		1	5	10	10	5	1			
6		1	6	15	20	15	6	1		
7		1	7	21	35	35	21	7	1	
8		1	8	28	56	70	56	28	8	1

Each entry not equal to one is the sum of the numbers above and above and to the left. This can be explained quite easily: $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$. Suppose you have n unique objects: in how many ways can you choose k objects from those n?

1. If $k = 0$, there is only one way to choose no objects out of the n.
2. If $k = n$, there is only one way to choose all objects out of n.
3. If $k = 1$, there is exactly n ways of choosing one object out of n.
4. If $k = n-1$, there is, again, exactly n ways of choosing $n-1$ object out of n: that is, take all but one of the objects.

However, what is the general case? Why are there six ways of choosing two out of four objects? That is, why is $\binom{4}{2} = 6$? Well, suppose you have n objects, and you want to determine how you many ways there are of choosing k of those n objects. Because the objects are unique, pick one, and set it aside. Now, there are two options:

1. Out of the remaining $n-1$ objects, you can ask how many ways are there to choose k of them. That is, you ignore your chosen object.

2. Alternatively, out of the remaining $n-1$ objects, you can ask how many ways are there to choose $k-1$ of them. Then, you add your chosen object to these $k-1$ objects to create a collection of k objects chosen from the n.

This is an exhaustive description of all possibilities, so we have that $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$. Thus, we have the following recurrence relation:

$$\binom{n}{k} = \begin{cases} 0 & \text{If } k < 0 \text{ or } k > n. \\ 1 & \text{If } k = 0 \text{ or } k = n. \\ \binom{n-1}{k} + \binom{n-1}{k-1} & \text{If } 0 < k < n. \end{cases}$$

The first entry emphasizes that there are no ways of choosing either -3 objects out of five or 7 objects out of only five objects.

For example, the number of ways of choosing four integers from the six integers $1, 2, 3, 4, 5, 6$ is 15:

$$\begin{array}{ccccc} 1,2,3,4 & 1,2,3,5 & 1,2,3,6 & 1,2,4,5 & 1,2,4,6 \\ 1,2,5,6 & 1,3,4,5 & 1,3,4,6 & 1,3,5,6 & 1,4,5,6 \\ 2,3,4,5 & 2,3,4,6 & 2,3,5,6 & 2,4,5,6 & 3,4,5,6 \end{array}$$

We can find this by determining first choosing 6, and setting it aside. Next, we ask how many ways we can choose four integers from the five integers $1, 2, 3, 4, 5$:

$$\begin{array}{ccccc} 1,2,3,4 & 1,2,3,5 & 1,2,4,5 & 1,3,4,5 & 2,3,4,5 \end{array}$$

This is only a subset of five of the fifteen listed above. Next, we ask how many ways are there of selecting three objects out of five:

$$\begin{array}{ccccc} 1,2,3 & 1,2,4 & 1,2,5 & 1,3,4 & 1,3,5 \\ 1,4,5 & 2,3,4 & 2,3,5 & 2,4,5 & 3,4,5 \end{array}$$

Notice that if we add "6" to each of these selections of three objects, we get the remaining ten.

If you sum up the rows of the above triangle, you get that the first row equals 1, the second $1+1=2$, the third $1+2+1=4$, the fourth $1+3+3+1=8$, and the next is $1+4+6+4+1=16$. This may lead you to believe that $\sum k = 0^n \binom{n}{k} = 2^n$. Can we prove this?

Theorem 2. Given $n \geq 0$, $\sum k = 0^n \binom{n}{k} = 2^n$.

Proof. We will use a proof by induction.

When $n = 0$, there is only one way to choose zero items from zero items, and that is to choose none (or all), and thus, $\binom{0}{0} = 1$. This is reinforced by $\frac{n!}{k!(n-k)!} = \frac{0!}{0! \cdot 0!} = \frac{1}{1 \cdot 1} = 1$.

Next, let us assume it is true for an arbitrary n. In this case, we have

$$\sum_{k=0}^{n+1} \binom{n+1}{k}.$$

However, because $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$, we have

$$\sum_{k=0}^{n+1} \left(\binom{n}{k} + \binom{n}{k-1} \right)$$

Because n is finite, we have $\sum_{k=0}^{n}(a_n + b_n) = (\sum_{k=0}^{n} a_a) + (\sum_{k=0}^{n} b_n)$, so in this case, we have

$$\sum_{k=0}^{n+1} \binom{n+1}{k} = \sum_{k=0}^{n+1} \binom{n}{k} + \sum_{k=0}^{n+1} \binom{n}{k-1}.$$

Let us examine these two sums carefully:

$$\sum_{k=0}^{n+1} \binom{n}{k} = \left(\sum_{k=0}^{n} \binom{n}{k} \right) + \binom{n}{n+1},$$

however, there are no possible ways to choose $n+1$ objects out of n, so $\binom{n}{n+1} = 0$, so

$$\sum_{k=0}^{n+1} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} = 2^n,$$

by our assumption.

Similarly,

$$\sum_{k=0}^{n+1} \binom{n}{k-1} = \binom{n}{-1} + \sum_{k=1}^{n} \binom{n}{k},$$

and $\binom{n}{-1} = 0$ as there are no possible ways of choosing "-1" objects out of n. Thus, $\sum_{k=0}^{n+1} \binom{n}{k-1} = \sum_{k=1}^{n} \binom{n}{k} = 2^n$.

Therefore,

$$\sum_{k=0}^{n}(a_n + b_n) = \left(\sum_{k=0}^{n} a_a\right) + \left(\sum_{k=0}^{n} b_n\right) = 2^n + 2^n = 2(2^n) = 2^{n+1}.$$

Therefore, because the statement is true for $n = 0$,
and the truth of the statement for n implies the truth of the statement for $n+1$,
it follows by induction that the statement is true for all $n \geq 0$. $\square$

The following is a C++ implementation of a function that recursively calls itself to calculate $\binom{n}{k}$.

```
int binomial( int const n, int const k ) {
    if ( (k < 0) || (k > n) ) {
    //            ^----------------------- Read this as 'or'
        return 0;   // <--- the result is equal to '0'
    } else if ( (k == 0) || (k == n) ) {
    //                    ^--------------- Again, read this as 'or'
        return 1;   // <--- the result is equal to '1'
    } else {
        return binomial( n - 1, k ) + binomial( n - 1, k - 1 );
    }
}
```

12.7 Additional sections

Notice that every polynomial in x of degree less than or equal to 5 can be written by multiplying each of the following monic monomials by the appropriate coefficient

$$x^5, x^4, x^3, x^2, x \text{ and } 1,$$

and then adding them. There are six coefficients required, so if the polynomial $p(x) = 3.2x^4 - 4.8x^2 + 5.1x + 9.7$ then $a_5 = 0$, $a_4 = 3.2$, $a_3 = 0$, $a_2 = -4.8$, $a_1 = 5.1$ and $a_0 = 9.7$.

Question: Can every linear polynomial $a_1x + a_0$ be written with appropriate multiples of $x - 1$ and $x + 1$? We want to find coefficients c_{-1} and c_1 such that

$$a_1x + a_0 = c_{-1}(x - 1) + c_1(x + 1).$$

Let's try it out:

$$a_1x + a_0 = c_{-1}x - c_{-1} + c_1x + c_1.$$

Thus, we have that

$$a_1x = c_{-1}x + c_1x$$

and

$$a_0 = -c_{-1} + c_1.$$

Thus, this gives us two equations in two unknowns:

$$\begin{aligned} c_{-1} + c_1 &= a_1 \\ -c_{-1} + c_1 &= a_0 \end{aligned}$$

Add Equation 1 to Equation 2:

$$\begin{aligned} c_{-1} + c_1 &= a_1 \\ 2c_1 &= a_0 + a_1 \end{aligned}$$

Thus, $c_1 = \frac{a_0+a_1}{2}$. Substitute this into Equation 1, and we get

$$c_{-1} = \frac{a_0 + a_1}{2} = a_1.$$

Therefore, solving for c_{-1}, we have

$$c_{-1} = a_1 - \frac{a_0 + a_1}{2} = \frac{a_1 - a_0}{2}.$$

Let's try this out: Suppose the polynomial is $p(x) = -112x + 42$. The claim is that $c_1 = \frac{-112+42}{2} = -35$ and $c_{-1} = \frac{-112-42}{2} = -77$; and behold that

$$-77(x-1) + (-35)(x+1) = -112x + 42.$$

13 Rationalizing the denominator

Given $x+r$, if you multiply this by $x-r$, you get x^2-r^2, with no intermediate term containing just x or r alone. We can use this to eliminate a radical: Suppose you are given $2-4\sqrt{5}$. This contains a radical, but if you multiply this by $2+4\sqrt{5}$, you get $2^2-(4\sqrt{5})^2 = 4-80 = -76$.

Therefore, if you see an expression like $\frac{5}{7-3\sqrt{2}}$ or $\frac{5+\sqrt{2}}{7-3\sqrt{2}}$, it is difficult to visualize the magnitude of the denominator, so it would be nice if we could replace the denominator by an integer. We do this by multiplying each expression by 1, but in this case, we will use $1 = \frac{7+3\sqrt{2}}{7+3\sqrt{2}}$. The value $7+3\sqrt{2}$ is called the "radical conjugate" of $7-3\sqrt{2}$. Thus, we have:

$$\frac{5}{7-3\sqrt{2}} = \frac{5}{7-3\sqrt{2}} \cdot 1 = \frac{5}{7-3\sqrt{2}} \cdot \frac{7+3\sqrt{2}}{7+3\sqrt{2}} = \frac{35+15\sqrt{2}}{7^2-3^2\cdot 2} = \frac{35+15\sqrt{2}}{49-18} = \frac{35+15\sqrt{2}}{31}.$$

We quickly see this is approximately $1.1 + 0.5 \cdot 1.4$.

Similarly,

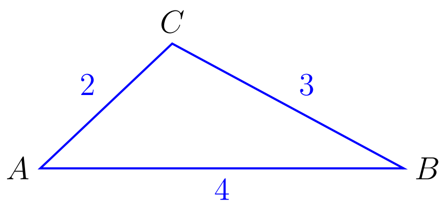
$$\frac{5+\sqrt{2}}{7-3\sqrt{2}} = \frac{(5+\sqrt{2})(7+3\sqrt{2})}{7^2-3^2\cdot 2} = \frac{35+(15+7)\sqrt{2}+3\cdot 2}{31} = \frac{41+22\sqrt{2}}{31}.$$

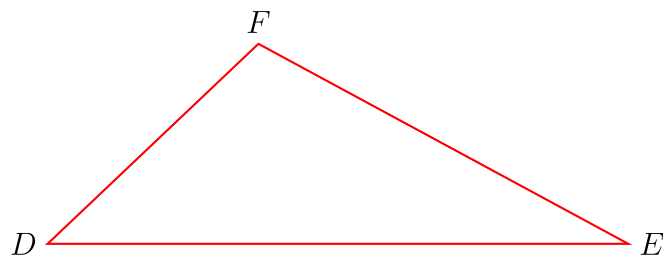
We quickly see this is approximately $1.3 + 0.7 \cdot 1.4$.

In general, if a, b and c are integers or rational numbers, then the radical conjugate of $a+b\sqrt{c}$ is $a-b\sqrt{c}$.

14 Similar triangles

To properly understand trigonometry, one must understand similar triangles. Two triangles ABC and DEF are said to be "similar" if they have the same angles. Here are two such triangles:





The angles at the three vertices are equal: $\angle A = \angle D$, $\angle B = \angle E$, and as the sum of the three angles equals 180° or π radians, so must the third: $\angle C = \angle F$. The length of the edges of the triangle on the left are $AB = 4$, $AC = 2$ and $BC = 3$. Thus, $BC = 1.5 \cdot AC$ and $AB = 2 \cdot AC$. The rule of similar triangles states therefore that in the second triangle, EF is 50% longer than DF, and DE is twice as long as DF.

This is often written as the relationship

$$\frac{AB}{AC} = \frac{DE}{DF}, \quad \frac{AB}{BC} = \frac{DE}{EF}, \text{ and } \quad \frac{AC}{BC} = \frac{DF}{EF}.$$

Thus, in this case, we know the ratios $\frac{AB}{AC} = \frac{4}{2} = 2$, $\frac{BC}{AB} = \frac{3}{4} = 0.75$, and $\frac{BC}{AC} = \frac{3}{2} = 1.5$,

What this means if that we know the length of one of the three sides of the second triangle, then we can always determine the lengths of the other two. In this case, suppose we know that $DE = 6.4$, and as $\frac{AB}{AC} = 2$, it follows that

$$\frac{DE}{DF} = 2 = \frac{6.4}{DF},$$

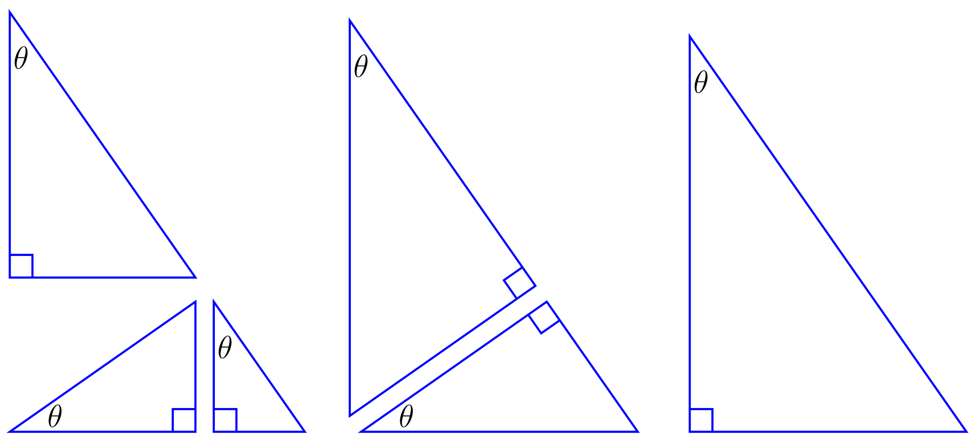
and therefore we can solve for DF to get that $DF = \frac{6.4}{2} = 3.2$. Similarly, as $\frac{BC}{AB} = 0.75$, it follows that

$$\frac{EF}{DE} = 0.75 = \frac{EF}{6.4},$$

and therefore we can solve for EF to get that $EF = 0.75 \cdot 6.4 = 4.8$.

For trigonometry, we will be focused entirely on right-angled triangles, and thus, if two right-angled triangles share a common angle θ, then the third angle must be $90° - \theta$ or $\pi - \theta$ radians, and therefore the triangles must be similar.

Specifically, there are six similar triangles we will ultimately look at:



In the three triangles on the left and the one on the right, the orientation of the similar sides remains unchanged, so one need only rotate the triangles to see the similarity. In the two triangles in the middle, they are similar to each other via a rotation, but to see the similarity with the other four, it is necessary to first reflect and then rotate.

15 Trigonometry

Here are easy ways to remember some of the easier formulas:

$$\sin(\,0^\circ\,) = \sin\,(\,0\,) = \tfrac{\sqrt{0}}{2} = 0.$$

$$\sin(30^\circ) = \sin\left(\tfrac{\pi}{6}\right) = \tfrac{\sqrt{1}}{2} = \tfrac{1}{2}.$$

$$\sin(45^\circ) = \sin\left(\tfrac{\pi}{4}\right) = \tfrac{\sqrt{2}}{2} = \tfrac{1}{\sqrt{2}}.$$

$$\sin(60^\circ) = \sin\left(\tfrac{\pi}{3}\right) = \tfrac{\sqrt{3}}{2}.$$

$$\sin(90^\circ) = \sin\left(\tfrac{\pi}{2}\right) = \tfrac{\sqrt{4}}{2} = 1.$$

$$\cos(\,0^\circ\,) = \cos\,(\,0\,) = \tfrac{\sqrt{4}}{2} = 1.$$

$$\cos(30^\circ) = \cos\left(\tfrac{\pi}{6}\right) = \tfrac{\sqrt{3}}{2}.$$

$$\cos(45^\circ) = \cos\left(\tfrac{\pi}{4}\right) = \tfrac{\sqrt{2}}{2} = \tfrac{1}{\sqrt{2}}.$$

$$\cos(60^\circ) = \cos\left(\tfrac{\pi}{3}\right) = \tfrac{\sqrt{1}}{2} = \tfrac{1}{2}.$$

$$\cos(90^\circ) = \cos\left(\tfrac{\pi}{2}\right) = \tfrac{\sqrt{0}}{2} = 0.$$

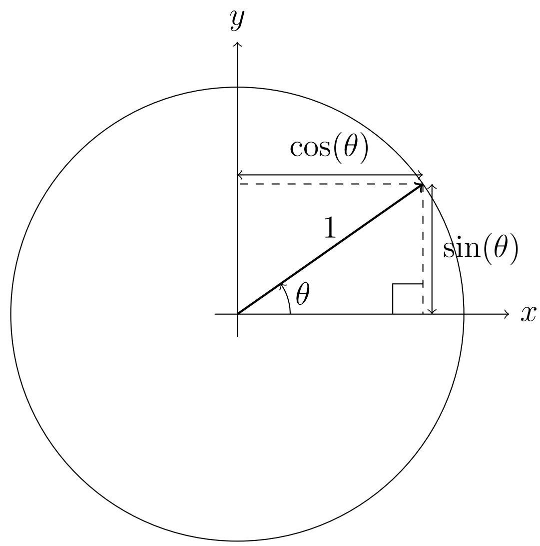
The angle is often represented by the variable θ, a Greek letter that is pronounced as "theta." Another common Greek letter used to represent angles is ϕ, pronounced "fie." Also, $(\sin(\theta))^2$ is often written as $\sin^2(\theta)$.

Here are a number of trigonometric formulas you should be aware of:

1. $\sin^2(\theta) + \cos^2(\theta) = 1.$
2. $\sin(-\theta) = -\sin(\theta)$, and thus $\sin(\theta) = -\sin(-\theta)$.
3. $\cos(-\theta) = \cos(\theta).$
4. $\cos(\theta) = \sin\left(\frac{\pi}{2} - \theta\right) = \sin\left(90^\circ - \theta\right).$
5. $\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right) = \cos\left(90^\circ - \theta\right).$
6. $\sin(2\theta) = 2\sin(\theta)\cos(\theta).$
7. $\cos(2\theta) = 2\cos^2(\theta) - 1.$
8. $\sin(\theta + \phi) = \sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi).$
9. $\cos(\theta + \phi) = \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi).$

Using the ratios of the last four, you can also determine the values of $\tan(2\theta)$ and $\tan(\theta+\phi)$. Note also that the last two simplify the two previous double-angle formulas if you let $\phi = \theta$.

You are already familiar with this image, showing a unit circle (a circle of radius 1):



First, where do the names come from?

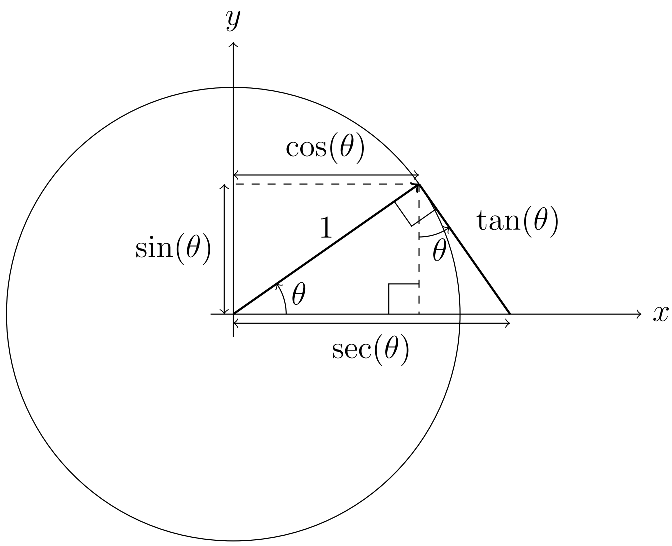
1. The term *sine* ultimately traces back to Sanskrit, where *jya* (or *ardha-jya*, meaning "chord" or "half-chord") was used. This was transliterated into Arabic as *jiba*, later written as *jaib*, meaning "fold" or "bosom." When medieval scholars translated Arabic texts into Latin, they rendered *jaib* as *sinus*, the Latin word for "fold" or "bay." This became the modern term *sine*. Coincidentally, the part of the toga that hangs from the shoulder is a fold, so you can always remember that the *sinus* is the part that hangs down, but you can also think of it as the height of the edge opposite the angle θ.

2. The term *cosine* is short for *complementary sine* (*sinus complementi* in Latin), since the cosine of an angle is the sine of its complementary angle. The cosine is the width of the edge opposite the complementary angle $90^\circ - \theta$.

   For example, to get to 90° (a right angle), if our angle is 30°, then to complete the right angle, the complement is 60°; and if our angle is 80°, then the complement is 10°. Thus, if the sine of 30° is $\frac{1}{2}$, the sine of the complement of 30° is $\frac{\sqrt{3}}{2}$.

From the Pythagorean theorem, from this, we may deduce that $\sin^2(\theta) + \cos^2(\theta) = 1$.

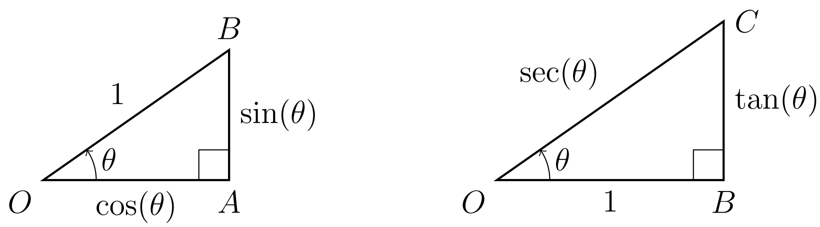
Now, you may have been blindly told that $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$. However, there is a geometric interpretation: if you draw a line that is *tangent* to the unit circle at the point $(\cos(\theta), \sin(\theta))$, then the distance between that point and the x-intercept of that tangent line is by definition $\tan(\theta)$, as shown here:



From this second right-angled triangle, we may deduce by using Pythagoras again that $1^2 + \tan^2(\theta) = \sec^2(\theta)$.

A secant line is any line that intersects a circle at two points. In the unit circle, the x-axis itself is a secant line, meeting the circle at $(-1, 0)$ and $(1, 0)$. In our construction, the line through the origin and the x-intercept of the tangent at $(\cos\theta, \sin\theta)$ also acts as a secant line. The length of the segment from the origin to this intercept, $(\sec(\theta), 0)$, is therefore called the secant of θ.

Let us show the two similar triangles next to each other:



Because we have similar triangles, that means that the ratio between corresponding sides must be equal. Thus, we have

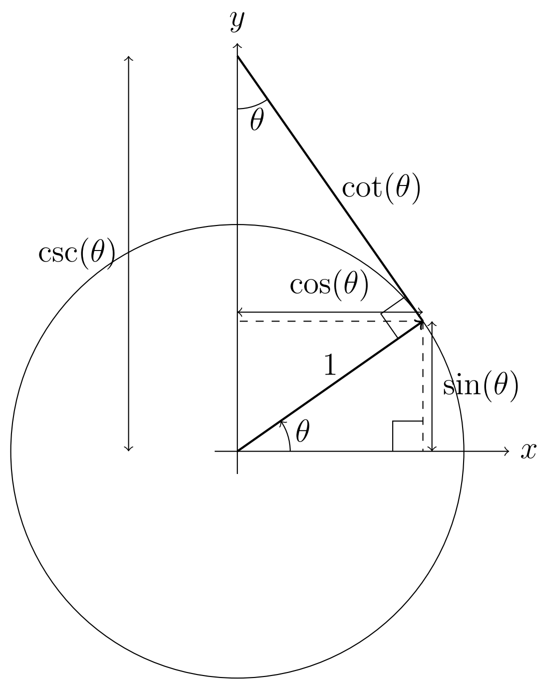
$$\frac{AB}{OA} = \frac{BC}{OB}, \quad \frac{OB}{OA} = \frac{OC}{OB} \quad \text{and} \quad \frac{AB}{OB} = \frac{BC}{OC}.$$

From this, we get that

$$\frac{1}{\cos(\theta)} = \frac{\sec(\theta)}{1}, \quad \frac{\sin(\theta)}{\cos(\theta)} = \frac{\tan(\theta)}{1}, \quad \text{and} \quad \frac{\sin(\theta)}{1} = \frac{\tan(\theta)}{\sec(\theta)}.$$

In other words, $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$ and $\sec(\theta) = \frac{1}{\cos(\theta)}$, not by definition, but rather as a consequence of how these two lengths are defined geometrically.

Just as tangent and secant are defined using a tangent line and a secant line, we can use the same construction with the complementary angle $90° - \theta$. In this case, the tangent line at $(\cos\theta, \sin\theta)$ intersects the y-axis. The vertical distance from the origin to this intercept, $(0, \csc(\theta))$, is called the cosecant of θ, while the horizontal distance along the tangent from the point of tangency to the y-axis is called the cotangent of θ. These lengths correspond to $\csc(\theta)$ and $\cot(\theta)$, respectively.

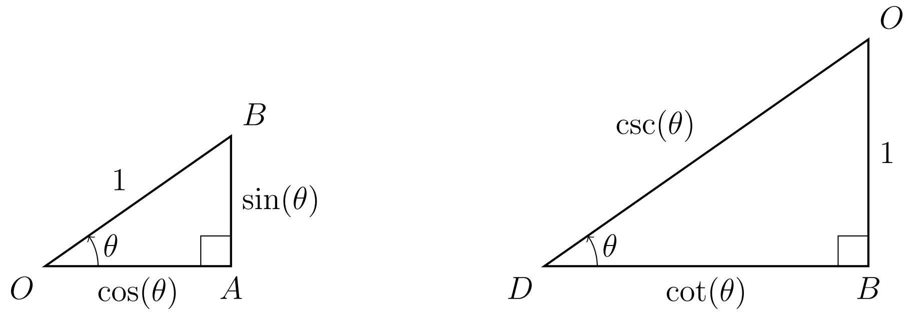


From this third right-angled triangle, we may deduce by using Pythagoras again that $1^2 + \cot^2(\theta) = \csc^2(\theta)$.

Similarly, in this diagram, we see the tangent of the complementary angle, so cotangent or $\cot(\theta)$ and the secant of the complementary angle, so cosecant or $\csc(\theta)$. Thus, because the complementary angle is always $90° - \theta$ or $\frac{\pi}{2} - \theta$, we have these six relationships:

1. $\sin(\theta) = \cos(90° - \theta)$
2. $\cos(\theta) = \sin(90° - \theta)$
3. $\tan(\theta) = \cot(90° - \theta)$
4. $\cot(\theta) = \tan(90° - \theta)$
5. $\sec(\theta) = \csc(90° - \theta)$
6. $\csc(\theta) = \sec(90° - \theta)$

Let us show the two similar triangles next to each other:



Again, because we have similar triangles, that means that the ratio between corresponding sides must be equal. Thus, we have

$$\frac{1}{\sin(\theta)} = \frac{\csc(\theta)}{1}, \quad \frac{\cos(\theta)}{\sin(\theta)} = \frac{\cot(\theta)}{1}, \quad \text{and} \quad \frac{\cos(\theta)}{1} = \frac{\cot(\theta)}{\csc(\theta)}.$$

In other words, $\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$ and $\csc(\theta) = \frac{1}{\sin(\theta)}$, again, not by definition, but rather as a consequence of how these two lengths are defined geometrically.

We now see that all four of these functions arise naturally from geometry on the unit circle.

- The *tangent* of θ is the horizontal distance along the tangent line at $(\cos\theta, \sin\theta)$, while the *cotangent* is the vertical distance along the tangent line to the y-axis.
- The *secant* of θ is the distance from the origin to the x-intercept of that tangent line, while the *cosecant* is the distance from the origin to the y-intercept of the same tangent line.

Remark

In this way, tangent and cotangent form a natural pair, and secant and cosecant form a natural pair. Moreover, each is related to sine or cosine through ratios or reciprocals:

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}, \quad \cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}, \quad \sec(\theta) = \frac{1}{\cos(\theta)}, \quad \csc(\theta) = \frac{1}{\sin(\theta)}.$$

Thus, the names are not arbitrary but come directly from the geometry of tangents and secants to the unit circle.

Finally, we may look at similar triangles between the second and third right-angled triangles, to get the similar relationships:

$$\frac{\sec(\theta)}{\tan(\theta)} = \frac{\csc(\theta)}{1}, \quad \frac{1}{\tan(\theta)} = \frac{\cot(\theta)}{1}, \quad \text{and} \quad \frac{1}{\sec(\theta)} = \frac{\cot(\theta)}{\csc(\theta)}.$$

From this, we know that $\cot(\theta) = \frac{1}{\tan(\theta)}$.

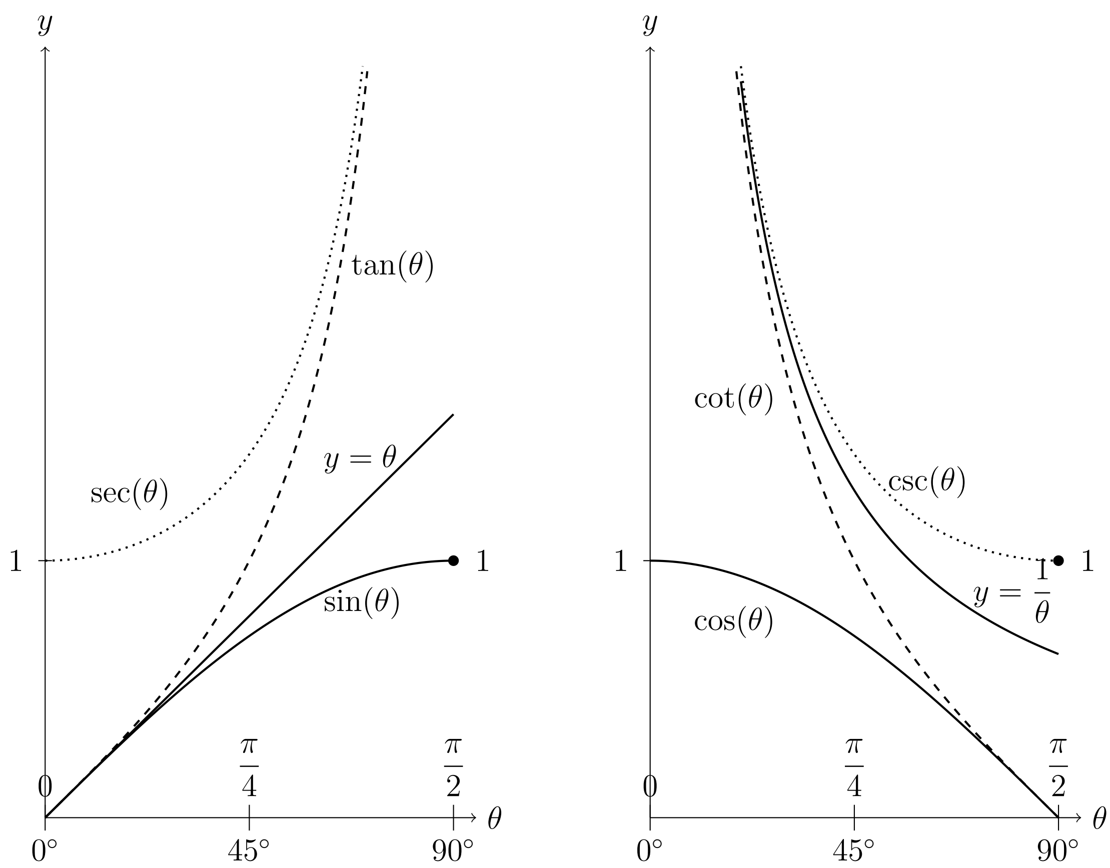
This last property leads to an interesting observation: the lengths of the two tangent lines ($\tan(\theta)$ and $\cot(\theta)$) must always have a product equal to 1.

You also have inequalities: for $0 < \theta < 90°$, $\sin(\theta) < \tan(\theta) < \sec(\theta)$ and $\cos(\theta) < \cot(\theta) < \csc(\theta)$. If you are using radians, then you'll see that:

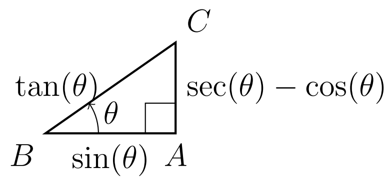
1. $\sin(\theta) \lesssim \theta \lesssim \tan(\theta)$ for $\theta \gtrsim 0$.
2. $\tan(\theta) \lesssim \frac{1}{\frac{\pi}{2}-\theta} \lesssim \sec(\theta)$ for $\theta \lesssim \frac{\pi}{2}$.
3. $\cot(\theta) \lesssim \frac{1}{\theta} \lesssim \csc(\theta)$ for $\theta \gtrsim 0$.
4. $\cos(\theta) \lesssim \frac{\pi}{2} - \theta \lesssim \csc(\theta)$ for $\theta \lesssim \frac{\pi}{2}$.

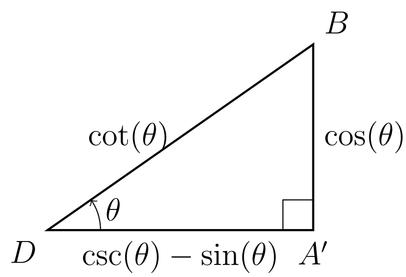
The symbol $x \lesssim y$ means that x is less than but approximately equal to y. Thus, again, if we are using radians (not degrees):

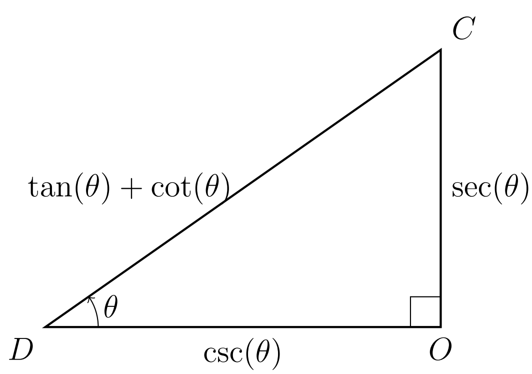
1. For very small values of θ,
2. $\sin(\theta) \approx \theta \approx \tan(\theta)$, and $\cot(\theta) \approx \dfrac{1}{\theta} \approx \csc(\theta)$.
3. For θ close to $\frac{\pi}{2}$, $\tan(\theta) \approx \dfrac{1}{\frac{\pi}{2} - \theta} \approx \sec(\theta)$, and $\cos(\theta) \approx \frac{\pi}{2} - \theta \approx \cot(\theta)$.



What you also may have noticed is that there are three more similar triangles in the diagrams above, ones not normally identified by your secondary school teachers:







From these images, it is clear that

1. $\sin^2(\theta) + (\sec(\theta) - \cos(\theta))^2 = \tan^2(\theta)$,
2. $(\csc(\theta) - \sin(\theta))^2 + \cos^2(\theta) = \cot^2(\theta)$, and
3. $\csc^2(\theta) + \sec^2(\theta) = (\tan(\theta) + \cot(\theta))^2$.

Can you show these are true by converting everything into sines and cosines and simplifying?

Similarly, taking the ratio of the corresponding sides, we also get that:

1. $\frac{\tan(\theta)}{\sin(\theta)} = \frac{\cot(\theta)}{\csc(\theta)-\sin(\theta)} = \frac{\tan(\theta)+\cot(\theta)}{\csc(\theta)}$
2. $\frac{\sec(\theta)-\cos(\theta)}{\sin(\theta)} = \frac{\cos(\theta)}{\csc(\theta)-\sin(\theta)} = \frac{\sec(\theta)}{\csc(\theta)}$
3. $\frac{\tan(\theta)}{\sec(\theta)-\cos(\theta)} = \frac{\cot(\theta)}{\cos(\theta)} = \frac{\tan(\theta)+\cot(\theta)}{\sec(\theta)}$

Again, can you show these are true in a similar manner? For example, converting everything in the first equation to sines and cosines, we have

$$\frac{\sin(\theta)}{\cos(\theta)\sin(\theta)} = \frac{\cos(\theta)}{\sin(\theta)(\frac{1}{\sin(\theta)} - \sin(\theta))} = \sin(\theta)\left(\frac{\sin(\theta)}{\cos(\theta)} + \frac{\cos(\theta)}{\sin(\theta)}\right),$$

and all these (with a little effort) simplify to $\frac{1}{\cos(\theta)}$.

Finally, because all the complementary functions refer to the same operation, but for the complementary angle, so $90^\circ - \theta$ or $\frac{\pi}{2} - \theta$, it follows that all of the functions can be written in terms of any one of them. For example, we can write all of them in terms of the sine function:

1. $\cos(\theta) = \sin(90^\circ - \theta) = \sin\left(\frac{\pi}{2} - \theta\right)$.
2. $\tan(\theta) = \frac{\sin(\theta)}{\sin(90^\circ - \theta)} = \frac{\sin(\theta)}{\sin\left(\frac{\pi}{2} - \theta\right)}$.
3. $\sec(\theta) = \frac{1}{\sin(90^\circ - \theta)} = \frac{1}{\sin\left(\frac{\pi}{2} - \theta\right)}$.
4. $\csc(\theta) = \frac{1}{\sin(\theta)}$.
5. $\cot(\theta) = \frac{\sin(90^\circ - \theta)}{\sin(\theta)} = \frac{\sin\left(\frac{\pi}{2} - \theta\right)}{\sin(\theta)}$.

These equivalences explain why some libraries implement only a limited set of trigonometric functions: you are expected to use identities to derive the others. Historically, some even provided only one such function (usually sin or more rarely tan). Also, some libraries for embedded systems only calculated these functions on the range $[0, 90^\circ]$ or $\left[0, \frac{\pi}{2}\right]$, requiring you to use additional identities to map your θ onto this range: for example

1. $\cos(-x) = \cos(x)$ and $\sec(-x) = \sec(x)$, while $\sin(-x) = -\sin(x)$, $\tan(-x) = -\tan(x)$, while $\csc(-x) = -\csc(x)$, and $\cot(-x) = -\cot(x)$.
2. For an integer n,
   (a) In all cases $\mathrm{trig}(\theta + 2\pi n) = \mathrm{trig}(\theta)$.
   (b) Also, $\tan(\theta + \pi n) = \tan(\theta)$ and $\cot(\theta + \pi n) = \cot(\theta)$.

(c) $\sin(x+(2n+1)\pi) = -\sin(x)$, $\cos(x+(2n+1)\pi) = -\cos(x)$, $\sec(x+(2n+1)\pi) = -\sec(x)$, and $\csc(x+(2n+1)\pi) = -\csc(x)$.

Remember, though, as an engineer, its more important to be aware that such identities exist, and you don't necessarily have to memorize them.

Remark

You will note that I use both degrees and radians. This is because different fields of engineering adopt whichever unit is most convenient. In power engineering, for example, degrees are preferred since phase separations are often thirds or sixths of a circle, making 120° and 60° more practical than $\frac{2\pi}{3}$ and $\frac{\pi}{3}$, respectively. In most other fields, however, radians are standard, as they arise naturally in calculus and make many fundamental formulas simpler. This is especially true in fields of engineering related to modeling with differential equations, for if θ is in radians, we have $\frac{\mathrm{d}}{\mathrm{d}\theta}\sin(\theta) = \cos(\theta)$, whereas if θ were measured in degrees, an additional conversion factor, $\frac{\mathrm{d}}{\mathrm{d}\theta}\sin(\theta) = \frac{\pi}{180^\circ}\cos(\theta)$, would always be required.

15.1 For entertainment

You don't have to know these:

1. $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
2. $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$
3. $\tan(2\theta) = \frac{2\tan(\theta)}{1-\tan^2(\theta)}$
4. $\cot(2\theta) = \frac{\cot(\theta)-1}{2\cot(\theta)}$
5. $\sec(2\theta) = \frac{1}{\cos^2(\theta)-\sin^2(\theta)}$
6. $\csc(2\theta) = \frac{1}{2\sin(\theta)\cos(\theta)}$

With these, the sign is chosen depending on the quadrant of θ:

1. $\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{2}}$
2. $\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos(\theta)}{2}}$
3. $\tan\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{1+\cos(\theta)}}$
4. $\cot\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos(\theta)}{1-\cos(\theta)}}$
5. $\sec\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{2}{1+\cos(\theta)}}$
6. $\csc\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{2}{1-\cos(\theta)}}$

15.2 More fun

Observe that $\sin^2(\theta) + \cos^2(\theta) = 1$. For each of the other triangles, we can quickly deduce the Pythagorean theorem:

1. Divide both sides by $\cos^2(\theta)$ to get $\tan^2(\theta) + 1 = \sec^2(\theta)$.
2. Divide both sides by $\sin^2(\theta)$ to get $1 + \cot^2(\theta) = \csc^2(\theta)$.
3. Divide both sides by $\sin^2(\theta)\cos^2(\theta)$ to get $\sec^2(\theta) + \csc^2(\theta) = \frac{1}{\sin^2(\theta)\cos^2(\theta)}$.
   You will observe that $(\tan(\theta) + \cot(\theta))^2 = \left(\frac{\sin(\theta)}{\cos(\theta)} + \frac{\cos(\theta)}{\sin(\theta)}\right)^2$.
   To get a common denominator, multiply both the first by $1 = \frac{\sin(\theta)}{\sin(\theta)}$ and the second by $\frac{\cos(\theta)}{\cos(\theta)}$ to get $(\tan(\theta) + \cot(\theta))^2 = \left(\frac{\sin^2(\theta)+\cos^2(\theta)}{\sin(\theta)\cos(\theta)}\right)^2 = \left(\frac{1}{\sin(\theta)\cos(\theta)}\right)^2 = \frac{1}{\sin^2(\theta)\cos^2(\theta)}$.
   Thus, we may conclude that $\sec^2(\theta) + \csc^2(\theta) = (\tan(\theta) + \cot(\theta))^2$.

As an exercise at home, show how we can get from $\sin^2(\theta) + \cos^2(\theta) = 1$ to each of the following:

1. $\sin^2(\theta) + (\sec(\theta) - \cos(\theta))^2 = \tan^2(\theta)$
2. $(\csc(\theta) - \sin(\theta))^2 + \cos^2(\theta) = \cot^2(\theta)$

16 Arithmetic sequences and sums, and geometric sequences, sums, and series

An arithmetic sequence is a sequence of values where each subsequent value is the previous plus a fixed difference d. Thus, given the first term in such a sum, a_0, we may therefore define the k^{th} term as

$$a_k = a_0 + kd.$$

We may want to calculate this sum up to $k = n$, in which case, we must calculate

$$\sum_{k=0}^{n} a_k = \sum_{k=0}^{n} (a_0 + kd).$$

But what does this equal? First, notice that if we were to add

$$(s_0 + t_0) + (s_1 + t_1) + (s_2 + t_2) + \cdots + (s_{n-1} + t_{n-1}) + (s_n + t_n),$$

we could calculate this sum by first calculating these two sums:

$$S = s_0 + s_1 + s_2 + \cdots + s_{n-1} + s_n$$

and

$$T = t_0 + t_1 + t_2 + \cdots + t_{n-1} + t_n,$$

and then calculating $S + T$. This is the same as saying

$$\sum_{k=0}^{n}(s_k + t_k) = \left(\sum_{k=0}^{n} s_k\right) + \left(\sum_{k=0}^{n} t_k\right).$$

Thus, we can rewrite this arithmetic sum as

$$\sum_{k=0}^{n} a_k = \sum_{k=0}^{n} (a_0 + kd) = \left(\sum_{k=0}^{n} a_0\right) + \left(\sum_{k=0}^{n} kd\right).$$

Second, notice that if we were to add a sequence of values where each value has a common multiplier of c, such as

$$cu_0 + cu_1 + cu_2 + \cdots + cu_{n-1} + cu_n,$$

then we could factor out the c, as follows:

$$c(u_0 + u_1 + u_2 + \cdots + u_{n-1} + u_n),$$

and therefore we see that

$$\sum_{k=0}^{n}(cu_k) = c\sum_{k=0}^{n} u_k.$$

In the first sum, the common multiplier is a_0, and in the second it is d, so we now have

$$\sum_{k=0}^{n} a_k = a_0\left(\sum_{k=0}^{n} 1\right) + d\left(\sum_{k=0}^{n} k\right).$$

In case it looks a little odd, the first sum is just $a_0 + a_0 + \cdots + a_0$, which equals $a_0(1 + 1 + \cdots + 1)$. The next question is, what are these sums

$$\sum_{k=0}^{n} 1 \quad \text{and} \quad \sum_{k=0}^{n} k$$

equal to? Because we are starting at 0, the first sum adds 1 a total of $n + 1$ times, so

$$\sum_{k=0}^{n} 1 = n + 1.$$

The second sum, because it is so intimately attached to arithmetic sums, is called "the arithmetic sum". How can we deduce a formula for this sum? First, we can look at the partial sums:

$$0, 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, \ldots.$$

If we have $n = 7$, we are calculating $1+2+3+4+5+6+7$. This is equivalent to determining how many stars appear in the triangle

```
*
* *
* * *
* * * *
* * * * *
* * * * * *
* * * * * * *
```

Now, the total number of points in a 7×7 square is 49:

```
--- * o o o o o o
 ^  * * o o o o o
 |  * * * o o o o
 7  * * * * o o o
 |  * * * * * o o
 v  * * * * * * o
--- * * * * * * *
    |<--- 7 --->|
```

This is approximately half of 49, so we could start by comparing the actual sum versus $\frac{n^2}{2}$. In this table, we proceed as follows: The first column is n. The next column is $\frac{n^2}{2}$, our approximation. The next column contains the actual values of the sums (calculated by hand). Finally, the last value is the difference between these two.

n	$\frac{n^2}{2}$	Actual value	Difference
0	0	0	0
1	0.5	1	0.5
2	2	3	1
3	4.5	6	1.5
4	8	10	2
5	12.5	15	2.5
6	18	21	3
7	24.5	28	3.5
8	32	36	4
9	40.5	45	4.5
10	50	55	5

Table 1: Approximating an arithmetic series

You will notice that the difference is always $\frac{n}{2}$, and therefore it seems that the correct answer is not $\frac{n^2}{2}$, but rather $\frac{n^2}{2} + \frac{n}{2}$. We can rewrite this formula so that it may be more familiar:

$$\frac{n^2}{2} + \frac{n}{2} = \frac{n^2 + n}{2} = \frac{n(n+1)}{2}.$$

In secondary school, you were given the formula. Above is one method to try to find this formula if you did not know the answer. Another appraoch might be to re-examine the image:

```
--- *
 ^  * *
 |  * * *
 7  * * * *
 |  * * * * *
 v  * * * * * *
--- * * * * * * *
    |<--- 7 --->|
```

Let us draw this triangle again, but reflected in both axes and then shifted to the right:

```
--- * x x x x x x x
 ^  * * x x x x x x
 |  * * * x x x x x
 7  * * * * x x x x
 |  * * * * * x x x
 v  * * * * * * x x
--- * * * * * * * x
    |<--- 7 --->|
```

This adds one more column, and so twice the number of points is $7 \cdot 8$, and therefore the number of asterisks is exactly half this $\frac{7\times 8}{2}$. Looking at this, as $n = 7$, one might think that the actual formula is now $\frac{n(n+1)}{2}$; the same formula we found above. However, we must ask, is this formula correct? If you were to try it for different values, you would see that it looks like it is indeed correct, but we haven't tried to for $n = 1000$ or $n = 1000000000$. We cannot try every possible integer, so instead, we will rely on a proof.

First, the formula gives us the correct answer when $n = 0$, for $\frac{0\cdot(0+1)}{2} = 0$. We will call this our "base case".

Next, let us look at the sum

$$0 + 1 + 2 + \cdots + (n-2) + (n-1) + n + (n+1),$$

or

$$\sum_{k=0}^{n+1} k.$$

According to our formula, the sum should be $\frac{(n+1)((n+1)+1)}{2} = \frac{(n+1)(n+2)}{2}$. However, also notice that we can rewrite the sum as follows:

$$[0 + 1 + 2 + \cdots + (n-2) + (n-1) + n] + (n+1),$$

This is the sum

$$\left(\sum_{k=0}^{n} k\right) + (n+1).$$

If the formula is correct for n, we should be able to replace the sum with the formula for n:

$$\frac{n(n+1)}{2} + (n+1).$$

We may now use algebra as follows:

$$\begin{aligned}\frac{n(n+1)}{2} + (n+1) &= \frac{n(n+1)}{2} + \frac{2(n+1)}{2} \\ &= \frac{n(n+1) + 2(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2}\end{aligned}$$

This is indeed the expected formula with $n+1$, as described above.

Thus, if the formula is correct for n, it is also correct for $n+1$.

We may start at our base case:

1. The formula is correct for $n = 0$, as $\frac{0\cdot 1}{2} = 0$.

2. If the formula is correct for $n = 0$, it is correct for $n = 1$.

3. If the formula is correct for $n = 1$, it is correct for $n = 2$.
4. If the formula is correct for $n = 2$, it is correct for $n = 3$.
5. If the formula is correct for $n = 3$, it is correct for $n = 4$.
6. If the formula is correct for $n = 4$, it is correct for $n = 5$.

Notice that we could go on forever: this formula must be correct for all non-negative integers.

What we have shown above is called a "proof by induction". This is where you show a formula is true for a base case, and then show that if the formual is true for n, then it is also true for $n+1$. Thus, we have shown both of these, and therefore the formula must be correct for all non-negative n.

Remark

Here is an alternative problem: $n^3 - n$ is always divisible by three.
We can try this out for $n = 10$, so 990 is divisible by three. Also, for $n = 11$, $1331 - 11 = 1320$ is divisible by three. Remember that a number is divisible by three if and only if the sum of the digits is divisible by three, and $1+3+2+0 = 6$ is divisible by three, and therefore 1320 is divisible by three, as well.
When $n = 0$, $0^3 - 0 = 0$.
Let us assume that $n^3 - n$ is divisible by three.
In this case, our question is: Is $(n+1)^3 - (n+1)$ divisible by three?
To proceed, we expand this formula out by using the binomial coefficients described above:

$$\begin{aligned}(n+1)^3 - (n+1) &= n^3 + 3n^2 + 3n + 1 - n - 1 \\ &= (n^3 - n) + 3n^2 + 3n + 1 - 1 \\ &= (n^3 - n) + 3n^2 + 3n \\ &= (n^3 - n) + 3(n^2 + n)\end{aligned}$$

Looking at this, we see that $3(n^2 + n)$ has a factor of three, and is therefore divisible by three. Also, we previously assumed that the formula $n^3 - n$ is divisible by three, so the sum of two numbers, both of which are divisible by three, must also be divisible by three.
Therefore, $(n+1)^3 - (n+1)$ is divisible by three, and therefore, by the process of mathematical induction, all numbers of the form $n^3 - n$ must be divisible by three, even if $n = 534412$.

Therefore, we may conclude that

$$\sum_{k=0}^{n} (a_0 + kd) = (n+1)a_0 + d\frac{n(n+1)}{2}.$$

Remark

Let's look where a proof by induction can show a formula is wrong. Let us suppose someone came up to you and claimed that

$$\sum_{k=1}^{n} k = \frac{n^2+1}{2}.$$

That person could quickly show that the formula is correct when $n = 1$, for $\frac{1^2+1}{2} = \frac{2}{2} = 1$.
However, suppose it is true for n. Then according to the formula, it follows that

$$\sum_{k=1}^{n+1} k = \frac{(n+1)^2+1}{2} = \frac{n^2+2n+2}{2}.$$

But now we try to find the formula for $n+1$ using the formula for n:

$$\begin{aligned}
\sum_{k=1}^{n+1} k &= 1+2+3+\cdots+n+(n+1) \\
&= (1+2+3+\cdots+n)+(n+1) \\
&= \left(\sum_{k=1}^{n} k\right)+(n+1) \\
&= \frac{n^2+1}{2}+(n+1)
\end{aligned}$$

We got the last statement from substituting the formula when the upper bound is n. Now, however, we have a problem, because if we expand this expression, we get

$$\begin{aligned}
\sum_{k=1}^{n+1} k &= \frac{n^2+1}{2}+(n+1) \\
&= \frac{n^2+1}{2}+\frac{2(n+1)}{2} \\
&= \frac{n^2+1+2(n+1)}{2} \\
&= \frac{n^2+2n+3}{2}
\end{aligned}$$

However, when we substituted $n+1$ into the formula, we got $\frac{n^2+2n+2}{2}$, and these are not equal. Thus, the formula is wrong.
For example, if $n = 11$, the formula claims the sum is 61, and thus, if we sum to $n = 12$, we should get $61 + 12 = 73$, but the formula when we substitute in the value $n = 12$, we get $\frac{12^2+1}{2} = \frac{143}{2} = 72.5$.
Important: You do not need to use proof by induction to show that a formula is wrong. We simply present it here to show how we "could" use it to show a formula is wrong. To show a formula is wrong, the easiest way is to find a value for which the formula gives the wrong value: $1+2+3=6$, but the formula predicts $\frac{3^2+1}{2} = \frac{10}{2} = 5$.

Next, let us look at geometric sequences. A sequence is "geometric" if every subsequent term is a fixed multiple, or ratio, of the previous term. Thus, if the first value is a_0 and the ratio is r, we have the sequence

$$a_0, a_0r, a_0r^2, a_0r^3, a_0r^4, a_0r^5, a_0r^6, \ldots.$$

We would like to calculate these finite sums:

$$\sum_{k=0}^{n} a_0 r^k.$$

Using the rule we saw previously, we can factor out the common multiplier:

$$a_0 \sum_{k=0}^{n} r^k.$$

Thus, we are really only interested in finding the sum

$$\sum_{k=0}^{n} r^k.$$

Let us call this S_n, so

$$S_n = \sum_{k=0}^{n} r^k.$$

If we multiply S_n by r, we have:

$$rS_n = r\sum_{k=0}^{n} r^k = \sum_{k=0}^{n} r^{k+1}.$$

Thus, $rS_n = r + r^2 + \cdots + r^n + r^{n+1}$. We can therefore calculate the difference

$$S_n - rS_n = \left(1 + r + r^2 + \cdots + r^n\right) - \left(r + r^2 + \cdots + r^n + r^{n+1}\right)$$

Notice that each term from r to r^n in the first sum is canceled out by the corresponding term in the second sum, thus, we are left with

$$S_n - rS_n = 1 - r^{n+1},$$

and factoring the S_n on the left-hand side, we have

$$S_n(1 - r) = 1 - r^{n+1},$$

and thus, dividing both sides by $1 - r$, we have

$$S_n = \frac{1 - r^{n+1}}{1 - r}.$$

We can check this formula using $r = 1.2$ and $n = 5$, as

$$1 + 1.2 + 1.44 + 1.728 + 2.0736 + 2.48832 = 9.92992.$$

Similarly, calculating

$$\frac{1 - 1.2^6}{1 - 1.2} = \frac{1 - 2.985984}{-0.2} == \frac{-1.985984}{-0.2} = 9.92992.$$

This is the same answer we got above by adding the terms.

We can also prove that $\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$ by induction:

Proof. The formula is correct when $n = 0$, as $\sum_{k=0}^{0} r^k = r^0 = 1$ and this equals $\frac{1-r^{0+1}}{1-r} = 1$.

Next, we assume that the formula is correct for an arbitrary n, and then we note that

$$\begin{aligned}
\sum_{k=0}^{n+1} r^k &= 1 + r + r^2 + \cdots + r^n + r^{n+1} \\
&= \left(1 + r + r^2 + \cdots + r^n\right) + r^{n+1} \\
&= \left(\sum_{k=0}^{n} r^k\right) + r^{n+1} \\
&= \frac{1 - r^{n+1}}{1 - r} + r^{n+1} \\
&= \frac{1 - r^{n+1}}{1 - r} + \frac{(1 - r)r^{n+1}}{1 - r} \\
&= \frac{1 - r^{n+1}}{1 - r} + \frac{r^{n+1} - r^{n+2}}{1 - r} \\
&= \frac{1 - r^{n+1} + r^{n+1} - r^{n+2}}{1 - r} \\
&= \frac{1 - r^{n+2}}{1 - r}
\end{aligned}$$

This is indeed the formula we found above when we substite $n + 1$ into that formula. $\square$

The next question is, what is the infinite sum equal to, if it even has a value? Looking at

$$S_n = \frac{1 - r^{n+1}}{1 - r},$$

we note that if $|r| > 1$, then r^{n+1} continues to grow forever, so this can never converge to a single value.

If $r = 1$, the denominator is 0, so the ratio is undefined, but we note that if the ratio is 1, our sum is

$$\sum_{k=0}^{\infty} 1$$

and this grows to infinity.

If $r = -1$, the denominator is 2, so now our formula is

$$\frac{1 - (-1)^{n+1}}{2}.$$

This gives us the series of values

$$0, 1, 0, 1, 0, 1, 0, 1, 0, 1, \ldots,$$

and this sequence does not converge.

However, if $|r| < 1$, then as n becomes large $|r^n|$ becomes arbitrarily small. Thus, in your course on calculus, you will learn the tools to prove that if $|r| < 1$,

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}.$$

Therefore, the geometric sum $a_0 + a_0 r + a_0 r^2 + \cdots$ is, assuming $|r| < 1$, the value $\frac{a_0}{1-r}$.

If $r = 0.5$ and $a_0 = 1$, we see that

$$\sum_{k=0}^{\infty} 0.5^k = \frac{1}{1-0.5} = \frac{1}{0.5} = 2,$$

and this is what we intuitively understand to be true if we add

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots,$$

as the partial sums are

$$1, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \frac{31}{16}, \cdots,$$

which gets ever closer and closer to 2 without ever reaching it.

You will notice that if the ratio r is rational, then both the finite and infinite sums (assuming $|r| < 1$) are all rational numbers.

Remark

For the arithmetic sums, we calculated $\sum_{k=0}^{n} k^0$ and $\sum_{k=0}^{n} k^1$. Some of you may have seen formulas such as $\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ and $\sum_{k=0}^{n} k^3 = \frac{n^2(n+1)^2}{4}$. These formulas can become very ugly; however, there is one beautiful approximation:

$$\sum_{k=0}^{n} k^d \approx \frac{n^{d+1}}{d+1}.$$

You don't have to memorize this formula.

17 Rational numbers versus the real numbers

You can skip this section if you wish.

From the study of arithmetic and geometric sequences and series, we saw that the sum of a finite collection of rational numbers is always a rational number. We also observed that an infinite geometric sum of rational terms can converge to a rational limit, even though that limit never appears among the partial sums themselves. However, this is not always the case: a sequence of rational numbers may converge to a value that is not rational at all.

As a familiar example, recall the geometric series

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots .$$

Each partial sum is a rational number, and as more and more terms are added, the sequence of partial sums gets closer and closer to 2, which is itself rational. This shows that an infinite process involving rational numbers can still converge to a rational result. However, not every such limit is rational: there are infinite sums of rational numbers whose limit is irrational.

17.1 Approximating e

One striking example arises in the series

$$\sum_{k=0}^{\infty} \frac{1}{k!},$$

which converges to the number e.

Thus, we will consider these partial sums for various values of N:

$$\sum_{k=0}^{N} \frac{1}{k!} = \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \cdots + \frac{1}{N!}.$$

What happens if we try to keep adding values to infinity: $\sum_{k=0}^{\infty} \frac{1}{k!} = ?$. We can compute the partial sums, and they seem to converge to something like $2.718281828 \cdots$:

0	1	$= 1$
1	2	$= 2$
2	$\frac{5}{2}$	$= 2.5$
3	$\frac{8}{3}$	$= 2.\overline{6}$
4	$\frac{65}{24}$	$= 2.708\overline{3}$
5	$\frac{163}{60}$	$= 2.71\overline{6}$
6	$\frac{1957}{720}$	$= 2.7180\overline{5}$
7	$\frac{685}{252}$	$= 2.71\overline{825396}$
8	$\frac{109601}{40320}$	$= 2.7182787\overline{698412}$
9	$\frac{98641}{36288}$	$= 2.718281\overline{52557319223985890 6}$
10	$\frac{9864101}{3628800}$	$= 2.71828180\overline{114638447971781305}$
11	$\frac{13563139}{4989600}$	$= 2.7182818\overline{2619849286515953}$
12	$\frac{260412269}{95800320}$	$= 2.7182818282\overline{861685639463417241195018972796750574528352306130083907}$

Table 2: The series of partial sums $\sum_{k=0}^{n} \frac{1}{k!}$.

As you add more and more terms to this sum (so as N gets larger and larger), this approaches $2.71828182845904523536 0\cdots$, but it can be shown that this value is the irrational number we now call e.

17.2 Approximating $\sqrt{2}$

Here is another sequence of rational numbers that in the limit approaches an irrational number. Suppose that you have an approximation s of the square root of two, so $s \approx \sqrt{2}$. It then follows that:

1. If $s < \sqrt{2}$, then $\frac{2}{s} > \sqrt{2}$.
2. If $s > \sqrt{2}$, then $\frac{2}{s} < \sqrt{2}$.

In each case, you get the second relationship by multiplying both sides by $\frac{\sqrt{2}}{s}$. In either case, then, the average of these two should be closer to the square root of two than the original estimate s. Thus, given s, we calculate a better approximation using $\frac{1}{2}\left(s + \frac{2}{s}\right)$. Thus, starting with the horrible approximation $s = 1$ of the square root of two, we get the following sequence of approximations:

$$1$$

$$\frac{3}{2} = 1.5 \quad \left(= \frac{1}{2}\left(1 + \frac{2}{1}\right)\right)$$

$$\frac{17}{12} = 1.41\overline{6} \quad \left(= \frac{1}{2}\left(\frac{3}{2} + \frac{4}{3}\right)\right)$$

$$\frac{577}{408} = 1.414\overline{2156862745098039} \quad \left(= \frac{1}{2}\left(\frac{17}{12} + \frac{24}{17}\right)\right)$$

$$\frac{665857}{470832} = 1.4142\overline{135623746 \cdots \textit{570 more digits} \cdots 267}$$

$\frac{886731088897}{627013566048} = 1.4142135623730950488016896\cdots$

The correct answer, to 66 digits, is

$$\sqrt{2} \approx 1.41421356237309504880168872420969807856967187537694807317667973799\cdots,$$

however, an engineer will never need this much precision. In general, sixteen digits of precision is more than sufficient in most applications.

17.3 Approximating $\frac{\pi}{4}$

Finally, here is yet another sequence of rational numbers that converge to an irrational number:

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \cdots.$$

If you stop at any odd number in the denominator, the sum is a rational number, but the further you go, the closer this rational number comes to $\frac{\pi}{4}$.

17.4 Conclusions

Thus, as you are already aware, between any two rational numbers lie infinitely many other rationals, and between any two real numbers lie infinitely many other reals. However, we have now seen examples of sequences of rational numbers that get closer and closer to a "limit" which is not itself rational — such as e, $\sqrt{2}$, and π.

By contrast, the real numbers have the property of "completeness": whenever a sequence of real numbers converges to a limit, that limit is guaranteed to be a real number. It is not possible for a sequence of real numbers that converges to converge to anything other than another real number. This property is what makes the real number system richer and more powerful than the rationals.

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